MATH 311-504

Topics in Applied Mathematics

Lecture 2-5: Image and null-space (continued). General linear equations.

Image and null-space

Let V_1, V_2 be vector spaces and $f : V_1 \to V_2$ be a linear mapping.

V_1: the **domain** of f
V_2: the **range** of f

Definition. The **image** of f (denoted $\mathrm{Im}\, f$) is the set of all vectors $\mathbf{y} \in V_2$ such that $\mathbf{y} = f(\mathbf{x})$ for some $\mathbf{x} \in V_1$. The **null-space** of f (denoted $\mathrm{Null}\, f$) is the set of all vectors $\mathbf{x} \in V_1$ such that $f(\mathbf{x}) = \mathbf{0}$.

Theorem The image of f is a subspace of the range. The null-space of f is a subspace of the domain.

More examples

- $M : \mathcal{P} \to \mathcal{P}, \quad (Mp)(x) = xp(x).$

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$\implies (Mp)(x) = a_0 x + a_1 x^2 + \cdots + a_n x^{n+1}$

$\mathrm{Null}\, M = \{0\}, \quad \mathrm{Im}\, M = \{p(x) \in \mathcal{P} : p(0) = 0\}.$

- $I : \mathcal{P} \to \mathcal{P}, \quad (Ip)(x) = \int_0^x p(s)\, ds.$

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$\implies (Ip)(x) = a_0 x + \frac{1}{2} a_1 x^2 + \cdots + \frac{1}{n+1} a_n x^{n+1}$

$\mathrm{Null}\, I = \{0\}, \quad \mathrm{Im}\, I = \{p(x) \in \mathcal{P} : p(0) = 0\}.$

General linear equations

Definition. A **linear equation** is an equation of the form

$$\boxed{L(\mathbf{x}) = \mathbf{b},}$$

where $L : V \to W$ is a linear mapping, $\mathbf{b}$ is a given vector from W, and $\mathbf{x}$ is an unknown vector from V.

The image of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The null-space of L is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable then the general solution is

$$\mathbf{x}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k,$$

where $\mathbf{x}_0$ is a particular solution, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a spanning set for the null-space of L, and $t_1, \ldots, t_k$ are arbitrary scalars.

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Proof: Let $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$. Then

$$L(\mathbf{x}) = L(\mathbf{x}_0) + t_1 L(\mathbf{v}_1) + \cdots + t_k L(\mathbf{v}_k) = \mathbf{b}.$$

Conversely, if $L(\mathbf{x}) = \mathbf{b}$ then

$$L(\mathbf{x} - \mathbf{x}_0) = L(\mathbf{x}) - L(\mathbf{x}_0) = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Hence $\mathbf{x} - \mathbf{x}_0$ belongs to $\mathrm{Null}\, L$. It follows that $\mathbf{x} - \mathbf{x}_0 = t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$ for some $t_1, \ldots, t_k \in \mathbb{R}$.

Example. $\begin{cases} x+y+z=4, \\ x+2y=3. \end{cases}$

$$L:\mathbb{R}^3\to\mathbb{R}^2,\quad L\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&1&1\\1&2&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

Linear equation: $L(\mathbf{x})=\mathbf{b}$, where $\mathbf{b}=\begin{pmatrix}4\\3\end{pmatrix}$.

$$\left(\begin{array}{ccc|c}1&1&1&4\\1&2&0&3\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&4\\0&1&-1&-1\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&2&5\\0&1&-1&-1\end{array}\right)$$

$$\begin{cases} x+2z=5 \\ y-z=-1 \end{cases}\iff\begin{cases} x=5-2z \\ y=-1+z \end{cases}$$

$$(x,y,z)=(5-2t,-1+t,t)=(5,-1,0)+t(-2,1,1).$$

Example. $u''(x) + u(x) = e^{2x}$.

Linear operator $L : C^2(\mathbb{R}) \to C(\mathbb{R})$, $Lu = u'' + u$.
Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

It can be shown that the image of L is the entire space $C(\mathbb{R})$ while the null-space of L is spanned by the functions $\sin x$ and $\cos x$.

Observe that

$(Lb)(x) = b''(x) + b(x) = 4e^{2x} + e^{2x} = 5e^{2x} = 5b(x)$.

By linearity, $u_0 = \frac{1}{5}b$ is a particular solution.
Thus the general solution is

$$u(x) = \tfrac{1}{5}e^{2x} + t_1 \sin x + t_2 \cos x.$$

Let V_1, V_2 be vector spaces and $f : V_1 \to V_2$ be a linear mapping.

Definition. The map f is **one-to-one** if it maps different vectors from V_1 to different vectors in V_2. That is, for any $\mathbf{x}, \mathbf{y} \in V_1$ we have that

$$\mathbf{x} \neq \mathbf{y} \implies f(\mathbf{x}) \neq f(\mathbf{y}).$$

Theorem A linear mapping f is one-to-one if and only if $\operatorname{Null} f = \{\mathbf{0}\}$.

Proof: If a vector $\mathbf{x} \neq \mathbf{0}_1$ belongs to $\operatorname{Null} f$, then $f(\mathbf{x}) = \mathbf{0}_2 = f(\mathbf{0}_1) \implies f$ is not one-to-one.

On the other hand, if $\operatorname{Null} f$ is trivial then
$\mathbf{x} \neq \mathbf{y} \implies \mathbf{x} - \mathbf{y} \neq \mathbf{0} \implies f(\mathbf{x} - \mathbf{y}) \neq \mathbf{0}$
$\implies f(\mathbf{x}) - f(\mathbf{y}) \neq \mathbf{0} \implies f(\mathbf{x}) \neq f(\mathbf{y}).$

Let $f : V_1 \to V_2$ be a linear mapping.

Definition. The map f is **onto** if any vector from V_2 is the image under f of some vector from V_1. That is, if $\operatorname{Im} f = V_2$.

If the mapping f is both one-to-one and onto, then any $\mathbf{y} \in V_2$ is uniquely represented as $f(\mathbf{x})$, where $\mathbf{x} \in V_1$. In this case, we define the **inverse mapping** f^{-1} by $\boxed{f^{-1}(\mathbf{y}) = \mathbf{x} \iff f(\mathbf{x}) = \mathbf{y}.}$

If the mapping f is only one-to-one, we can still define the inverse mapping $f^{-1} : \operatorname{Im} f \to V_1$.

Theorem The inverse of a linear mapping is also linear.

Examples. • $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (x,y,x)$.
$\mathrm{Null}\, f = \{\mathbf{0}\}$, $\mathrm{Im}\, f$ is the plane $x = z$.
The inverse mapping $f^{-1} : \mathrm{Im}\, f \to \mathbb{R}^2$ is given by $(x,y,z) \mapsto (x,y)$.

• $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$.

$\det A = 1 \implies A$ is invertible.

g is one-to-one since

$$A\mathbf{x} = \mathbf{0} \implies \mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{0} = \mathbf{0}.$$

g is onto since $\mathbf{y} = A(A^{-1}\mathbf{y})$ for any $\mathbf{y} \in \mathbb{R}^2$.

The inverse mapping is given by $g^{-1}(\mathbf{y}) = A^{-1}\mathbf{y}$.