MATH 311-504

Topics in Applied Mathematics

Lecture 2-6:
Isomorphism.
Linear independence (revisited).

Definition. A mapping $f : V_1 \to V_2$ is **one-to-one** if it maps different elements from V_1 to different elements in V_2. The map f is **onto** if any element $y \in V_2$ is represented as $f(x)$ for some $x \in V_1$.

If the mapping f is both one-to-one and onto, then the inverse $f^{-1} : V_2 \to V_1$ is well defined.

Now let V_1, V_2 be vector spaces and $f : V_1 \to V_2$ be a linear mapping.

Theorem **(i)** The linear mapping f is one-to-one if and only if $\mathrm{Null}\, f = \{\mathbf{0}\}$.
(ii) The linear mapping f is onto if $\mathrm{Im}\, f = V_2$.
(iii) If the linear mapping f is both one-to-one and onto, then the inverse mapping f^{-1} is also linear.

Examples

- $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x, y, x)$.

$\mathrm{Null}\, f = \{\mathbf{0}\}$, $\mathrm{Im}\, f$ is the plane $x = z$.

The inverse mapping $f^{-1} : \mathrm{Im}\, f \to \mathbb{R}^2$ is given by $(x, y, z) \mapsto (x, y)$.

- $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$.

g is one-to-one and onto.

The inverse mapping is given by $g^{-1}(\mathbf{y}) = A^{-1}\mathbf{y}$.

- $L : \mathcal{P} \to \mathcal{P}, \ (Lp)(x) = p(x+1)$.

L is one-to-one and onto.
The inverse is given by $(L^{-1}p)(x) = p(x-1)$.

- $M : \mathcal{P} \to \mathcal{P}, \ (Mp)(x) = xp(x)$.

$\mathrm{Null}\, M = \{\mathbf{0}\}, \ \mathrm{Im}\, M = \{p(x) \in \mathcal{P} : p(0) = 0\}$.
The inverse mapping $M^{-1} : \mathrm{Im}\, M \to \mathcal{P}$ is given by
$(M^{-1}p)(x) = x^{-1}p(x)$.

- $I : \mathcal{P} \to \mathcal{P}, \ (Ip)(x) = \int_0^x p(s)\, ds$.

$\mathrm{Null}\, I = \{\mathbf{0}\}, \ \mathrm{Im}\, I = \{p(x) \in \mathcal{P} : p(0) = 0\}$.
The inverse mapping $I^{-1} : \mathrm{Im}\, I \to \mathcal{P}$ is given by
$(I^{-1}p)(x) = p'(x)$.

Isomorphism

Definition. A linear mapping $f : V_1 \to V_2$ is called an **isomorphism** of vector spaces if it is both one-to-one and onto.

Two vector spaces V_1 and V_2 are called **isomorphic** if there exists an isomorphism $f : V_1 \to V_2$.

The word "isomorphism" applies when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure, where "corresponding" means that the two parts play similar roles in their respective structures.

Examples of isomorphisms

- $\mathcal{M}_{2,2}(\mathbb{R})$ is isomorphic to $\mathbb{R}^4$.

Isomorphism: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)$.

- $\mathcal{M}_{2,3}(\mathbb{R})$ is isomorphic to $\mathcal{M}_{3,2}(\mathbb{R})$.

Isomorphism: $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$.

- The plane $z = 0$ in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.

Isomorphism: $(x, y, 0) \mapsto (x, y)$.

- $\mathcal{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$.

Isomorphism: $a_0 + a_1 x + \cdots + a_n x^n \mapsto (a_0, a_1, \ldots, a_n)$.

Classification problems of linear algebra

Problem 1 Given vector spaces V_1 and V_2, determine whether they are isomorphic or not.

Problem 2 Given a vector space V, determine whether V is isomorphic to $\mathbb{R}^n$ for some $n \geq 1$.

Problem 3 Show that vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not isomorphic if $m \neq n$.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

An infinite set $S \subset V$ is **linearly dependent** if there are some linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$. Otherwise S is **linearly independent**.

Theorem Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ are linearly dependent if and only if one of them is a linear combination of the other $k-1$ vectors.

Examples of linear independence

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ in $\mathbb{R}^3$.

$$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = \mathbf{0} \implies (x, y, z) = \mathbf{0}$$
$$\implies x = y = z = 0$$

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$$aE_{11} + bE_{12} + cE_{21} + dE_{22} = O \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O$$
$$\implies a = b = c = d = 0$$

Examples of linear independence

- Polynomials $1, x, x^2, \dots, x^n$.

$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ identically
$\implies \quad a_i = 0$ for $0 \le i \le n$

- The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$.

- Polynomials $p_1(x) = 1$, $p_2(x) = x - 1$, and $p_3(x) = (x-1)^2$.

$a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = a_1 + a_2(x-1) + a_3(x-1)^2 =$
$= (a_1 - a_2 + a_3) + (a_2 - 2a_3)x + a_3 x^2.$

Hence $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = 0$ identically
$\implies \quad a_1 - a_2 + a_3 = a_2 - 2a_3 = a_3 = 0$
$\implies \quad a_1 = a_2 = a_3 = 0$

Problem 1. Show that functions 1, e^x, and e^{-x} are linearly independent in $F(\mathbb{R})$.

Proof: Suppose that $a + be^x + ce^{-x} = 0$ for some $a, b, c \in \mathbb{R}$. We have to show that $a = b = c = 0$.

$x = 0 \implies a + b + c = 0$
$x = 1 \implies a + be + ce^{-1} = 0$
$x = -1 \implies a + be^{-1} + ce = 0$

The matrix of the system is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e & e^{-1} \\ 1 & e^{-1} & e \end{pmatrix}$.

$\det A = e^2 - e^{-2} - 2e + 2e^{-1} =$
$= (e - e^{-1})(e + e^{-1}) - 2(e - e^{-1}) =$
$= (e - e^{-1})(e + e^{-1} - 2) = (e - e^{-1})(e^{1/2} - e^{-1/2})^2 \neq 0.$

Hence the system has a unique solution $a = b = c = 0$.

Problem 2. Show that functions e^x, e^{2x}, and e^{3x} are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $ae^x + be^{2x} + ce^{3x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Differentiate this identity twice:

$$ae^x + 2be^{2x} + 3ce^{3x} = 0,$$
$$ae^x + 4be^{2x} + 9ce^{3x} = 0.$$

It follows that $A\mathbf{v} = \mathbf{0}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} ae^x \\ be^{2x} \\ ce^{3x} \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} ae^{x} \\ be^{2x} \\ ce^{3x} \end{pmatrix}.$$

To compute $\det A$, subtract the 1st row from the 2nd and the 3rd rows:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2.$$

Since A is invertible, we obtain

$$A\mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies ae^{x} = be^{2x} = ce^{3x} = 0$$
$$\implies a = b = c = 0$$

Problem 3. Show that functions x, e^x, and e^{-x} are linearly independent in $C(\mathbb{R})$.

Suppose that $ax + be^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Divide both sides of the identity by e^x:

$$axe^{-x} + b + ce^{-2x} = 0.$$

The left-hand side approaches b as $x \to +\infty$. $\implies b = 0$

Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the identity by x:

$$a + cx^{-1}e^{-x} = 0.$$

The left-hand side approaches a as $x \to +\infty$. $\implies a = 0$

Now $ce^{-x} = 0 \implies c = 0$.