MATH 311-504

Topics in Applied Mathematics

Lecture 2-8:
Basis and dimension.

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Equivalently, a subset $S \subset V$ is a basis for V if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \ldots, r_k \in \mathbb{R}$.

Examples. • Standard basis for $\mathbb{R}^n$:
$\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0)$,. . . ,
$\mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

• $n + 1$ polynomials $1, x, x^2, \dots, x^n$ form a basis for $\mathcal{P}_n = \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \;\mapsto\; \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

is a one-to-one correspondence between V and $\mathbb{R}^n$. This correspondence is linear (hence it is an isomorphism of V onto $\mathbb{R}^n$).

Vectors $\mathbf{v}_1=(2,5)$ and $\mathbf{v}_2=(1,3)$ form a basis for $\mathbb{R}^2$.

Problem 1. Find coordinates of the vector $\mathbf{v}=(3,4)$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$.

The desired coordinates x, y satisfy

$$\mathbf{v} = x\mathbf{v}_1 + y\mathbf{v}_2 \iff \begin{cases} 2x + y = 3 \\ 5x + 3y = 4 \end{cases} \iff \begin{cases} x = 5 \\ y = -7 \end{cases}$$

Problem 2. Find the vector $\mathbf{w}$ whose coordinates with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$ are $(3,4)$.

$$\mathbf{w} = 3\mathbf{v}_1 + 4\mathbf{v}_2 = 3(2,5) + 4(1,3) = (10,27)$$

The function $F(x) = \cosh(x+1)$ belongs to the vector space $W = \{f \in C^\infty \mid f'' - f = 0\}$.

Problem 1. Find coordinates of F with respect to the basis $\{e^x, e^{-x}\}$.

$$F(x) = \cosh(x{+}1) = \tfrac{1}{2}\big(e^{x+1} + e^{-(x+1)}\big) = \tfrac{e}{2}\, e^x + \tfrac{1}{2e}\, e^{-x}.$$

Problem 2. Find coordinates of F with respect to the basis $\{\cosh x,\ \sinh x\}$.

We have $F(x) = a\cosh x + b\sinh x$, where $a = F(0) = \cosh 1$, $b = F'(0) = \sinh 1$.

Bases for $\mathbb{R}^n$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$.

Theorem 1 If $m < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ do not span $\mathbb{R}^n$.

Theorem 2 If $m > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent.

Theorem 3 If $m = n$ then the following conditions are equivalent:

(i) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (as they are not parallel), but they do not span $\mathbb{R}^3$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-2) = 2 \neq 0.$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^3$ (because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ already span $\mathbb{R}^3$), but they are linearly dependent.

Dimension

Theorem Any vector space V has a basis. All bases for V are of the same cardinality.

Definition. The **dimension** of a vector space V, denoted $\dim V$, is the cardinality of its bases.

Remark. By definition, two sets are of the same cardinality if there exists a one-to-one correspondence between their elements.

For a finite set, the cardinality is the number of its elements.

For an infinite set, the cardinality is a more sophisticated notion. For example, $\mathbb{Z}$ and $\mathbb{R}$ are infinite sets of different cardinalities while $\mathbb{Z}$ and $\mathbb{Q}$ are infinite sets of the same cardinality.

Examples. • $\dim \mathbb{R}^n = n$

• $\mathcal{M}_{2,2}(\mathbb{R})$: the space of 2×2 matrices
$\dim \mathcal{M}_{2,2}(\mathbb{R}) = 4$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m\times n$ matrices
$\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

• $\mathcal{P}_n$: polynomials of degree at most n
$\dim \mathcal{P}_n = n + 1$

• $\mathcal{P}$: the space of all polynomials
$\dim \mathcal{P} = \infty$

• $\{\mathbf{0}\}$: the trivial vector space
$\dim \{\mathbf{0}\} = 0$

Classification problems of linear algebra

Theorem Two vector spaces are isomorphic if and only if they have the same dimension. In particular, a vector space V is isomorphic to $\mathbb{R}^n$ if and only if $\dim V = n$.

Example. Both $\mathcal{P}$ and $\mathbb{R}^\infty$ are infinite-dimensional vector spaces. However they are not isomorphic.

How to find a basis?

Theorem Let S be a subset of a vector space V. Then the following conditions are equivalent:

(i) S is a linearly independent spanning set for V, i.e., a basis;

(ii) S is a minimal spanning set for V;

(iii) S is a maximal linearly independent subset of V.

"Minimal spanning set" means "remove any element from this set, and it is no longer a spanning set".

"Maximal linearly independent subset" means "add any element of V to this set, and it will become linearly dependent".

Theorem Let V be a vector space. Then
(i) any spanning set for V can be reduced to a minimal spanning set;
(ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

Equivalently, any spanning set contains a basis, while any linearly independent set is contained in a basis.

Corollary A vector space is finite-dimensional if and only if it is spanned by a finite set.

How to find a basis?

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis.

Proposition Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k$ be a spanning set for a vector space V. If $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is also a spanning set for V.

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k =$$

$$= (t_0 r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0 r_k + t_k)\mathbf{v}_k.$$

How to find a basis?

Approach 2. Build a maximal linearly independent set adding one vector at a time.

If the vector space V is trivial, it has the empty basis.

If $V \neq \{\mathbf{0}\}$, pick any vector $\mathbf{v}_1 \neq \mathbf{0}$.
If $\mathbf{v}_1$ spans V, it is a basis. Otherwise pick any vector $\mathbf{v}_2 \in V$ that is not in the span of $\mathbf{v}_1$.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ span V, they constitute a basis. Otherwise pick any vector $\mathbf{v}_3 \in V$ that is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

And so on. . .