MATH 311-504

Topics in Applied Mathematics

Lecture 3-10:
Rotations in space.

Orthogonal matrices

Theorem Given an $n \times n$ matrix A, the following conditions are equivalent:

(i) A is orthogonal: $A^T = A^{-1}$;

(ii) columns of A form an orthonormal basis for $\mathbb{R}^n$;

(iii) rows of A form an orthonormal basis for $\mathbb{R}^n$;

(iv) $|A\mathbf{x}| = |\mathbf{x}|$ for any $\mathbf{x} \in \mathbb{R}^n$;

(v) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Orthogonal matrix is the transition matrix from one orthonormal basis to another.

Theorem If A is an orthogonal matrix then

- $\det A = 1$ or -1;
- each eigenvalue λ of A satisfies $|\lambda| = 1$.

Isometry

Definition. A transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry** if it preserves distance between points: $|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$.

Theorem Any isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and A is an orthogonal matrix.

Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n\times n$ orthogonal matrix.

Theorem There exists an orthonormal basis for $\mathbb{R}^n$ such that the matrix of L relative to this basis has a diagonal block structure

$$\begin{pmatrix} D_{\pm 1} & O & \dots & O \\ O & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & R_k \end{pmatrix},$$

where $D_{\pm 1}$ is a diagonal matrix whose diagonal entries are equal to 1 or -1, and

$$R_j = \begin{pmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{pmatrix}, \quad \phi_j \in \mathbb{R}.$$

Classification of 2×2 orthogonal matrices:

	$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
	rotation about the origin	reflection in a line
Determinant:	1	-1
Eigenvalues:	$e^{i\phi}$ and $e^{-i\phi}$	-1 and 1

Classification of 3×3 orthogonal matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

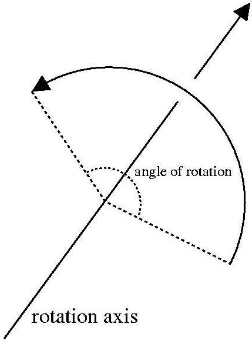
$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

$A =$ rotation about a line; $B =$ reflection in a plane; $C =$ rotation about a line combined with reflection in the orthogonal plane: $C = AB = BA$.

$\det A = 1$, $\det B = \det C = -1$.

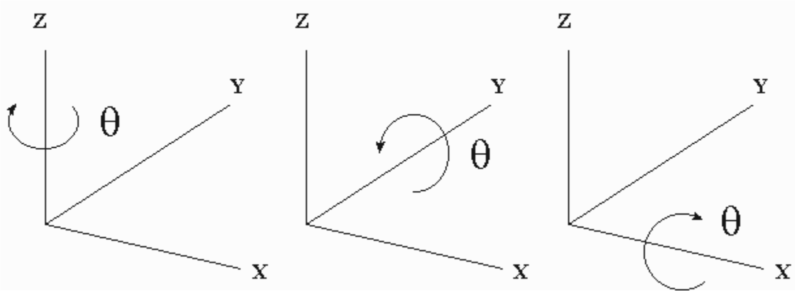
A has eigenvalues 1, $e^{i\phi}$, $e^{-i\phi}$. B has eigenvalues -1, 1, 1. C has eigenvalues -1, $e^{i\phi}$, $e^{-i\phi}$.

Rotations in space



If the axis of rotation is oriented, we can say about *clockwise* or *counterclockwise* rotations (with respect to the view from the positive semi-axis).

Clockwise rotations about coordinate axes



$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

Problem. Find the matrix of the rotation by $90^{\rm o}$ about the line spanned by the vector $\mathbf{a} = (1, 2, 2)$. The rotation is assumed to be counterclockwise when looking from the tip of **a**.

$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the matrix of (counterclockwise) rotation by $90^{\rm o}$ about the z-axis.

We need to find an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $\mathbf{v}_3$ has the same direction as **a**. Also, the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ should obey the same hand rule as the standard basis. Then B is the matrix of the given rotation relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let U denote the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (columns of U are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$). Then the desired matrix is $A = UBU^{-1}$.

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is going to be an orthonormal basis, the matrix U will be orthogonal. Then $U^{-1} = U^T$ and $A = UBU^T$.

Remark. The basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the same hand rule as the standard basis if and only if $\det U > 0$.

Hint. Vectors $\mathbf{a} = (1, 2, 2)$, $\mathbf{b} = (-2, -1, 2)$, and $\mathbf{c} = (2, -2, 1)$ are orthogonal.

We have $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 3$, hence $\mathbf{v}_1 = \frac{1}{3}\mathbf{b}$, $\mathbf{v}_2 = \frac{1}{3}\mathbf{c}$, $\mathbf{v}_3 = \frac{1}{3}\mathbf{a}$ is an orthonormal basis.

Transition matrix: $U = \frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$.

$$\det U = \frac{1}{27}\begin{vmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \frac{1}{27} \cdot 27 = 1.$$

(In the case $\det U = -1$, we should interchange vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.)

$$
\begin{aligned}
A &= UBU^T \\
&= \tfrac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \tfrac{1}{3}\begin{pmatrix} -2 & -1 & 2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{pmatrix} \\
&= \tfrac{1}{9}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 & 2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{pmatrix} \\
&= \tfrac{1}{9}\begin{pmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{pmatrix}.
\end{aligned}
$$

$U = \frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$ is an orthogonal matrix.

$\det U = 1 \implies U$ is a rotation matrix.

Problem. **(a)** Find the axis of the rotation.
(b) Find the angle of the rotation.

The axis is the set of points $\mathbf{x} \in \mathbb{R}^n$ such that $U\mathbf{x} = \mathbf{x} \iff (U - I)\mathbf{x} = \mathbf{0}$. To find the axis, we apply row reduction to the matrix $3(U - I)$:

$$3U - 3I = \begin{pmatrix} -5 & 2 & 1 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} -3 & 3 & 0 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -6 & 2 \\ 2 & 1 & -1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -6 & 2 \\ 0 & 3 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $U\mathbf{x} = \mathbf{x} \iff \begin{cases} x - z/3 = 0 \\ y - z/3 = 0 \end{cases}$

The general solution is $x = y = t/3,\ z = t,\ t \in \mathbb{R}$.
$\implies$ $\mathbf{d} = (1, 1, 3)$ is the direction of the axis.

$$U = \frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

Let ϕ be the angle of rotation. Then the eigenvalues of U are 1, $e^{i\phi}$, and $e^{-i\phi}$. Therefore

$$\det(U - \lambda I) = (1-\lambda)(e^{i\phi} - \lambda)(e^{-i\phi} - \lambda).$$

Besides, $\det(U - \lambda I) = -\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$, where $c_1 = \mathrm{Tr}\, U$ (the sum of diagonal entries). It follows that

$$\mathrm{Tr}\, U = 1 + e^{i\phi} + e^{-i\phi} = 1 + 2\cos\phi.$$

$$\mathrm{Tr}\, U = -2/3 \implies \cos\phi = -5/6 \implies \phi \approx 146.44^{\mathrm{o}}$$