MATH 311-504

Topics in Applied Mathematics

Lecture 3-11:
Fourier series.

Orthonormal bases

Standard basis for $\mathbb{R}^n$: $\mathbf{e}_1 = (1, 0, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$,... , $\mathbf{e}_n = (0, 0, 0, \dots, 1)$. It is an orthonormal system: $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ if $i \neq j$, $|\mathbf{e}_i| = 1$. For any $\mathbf{x} \in \mathbb{R}^n$ we have

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{x} \cdot \mathbf{e}_2)\mathbf{e}_2 + \cdots + (\mathbf{x} \cdot \mathbf{e}_n)\mathbf{e}_n.$$

Let V be an n-dimensional vector space endowed with an inner product. Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthonormal basis for V: $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $i \neq j$, while $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$. Then for any $\mathbf{x} \in V$ we have

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{x}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{x}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Orthogonal systems

Now suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is only an orthogonal basis for V. Then

$$\mathbf{x} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n.$$

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal set without $\mathbf{0}$ that is not a basis, then the right-hand side is the orthogonal projection of $\mathbf{x}$ onto $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Also, it is the best approximation of $\mathbf{x}$ by linear combinations $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ with respect to the distance

$$\mathrm{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\| = \sqrt{\langle \mathbf{y} - \mathbf{x}, \mathbf{y} - \mathbf{x} \rangle}.$$

Orthogonal polynomials

$\mathcal{P}$: the space of all polynomials.
Suppose that $\mathcal{P}$ is endowed with an inner product.
Let $P_0, P_1, P_2, \ldots$ be orthogonal polynomials relative to this inner product. Then for any $q \in \mathcal{P}$ we have

$$q(x) = \frac{\langle q, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) + \frac{\langle q, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1(x) + \cdots$$

The right-hand side is not really a series since $\langle q, P_n \rangle = 0$ for $n > \deg q$.

$$V = C[-\pi,\pi], \ \ \langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

$f_1(x) = \sin x, \ f_2(x) = \sin 2x, \ \dots, \ f_n(x) = \sin nx, \ \dots$

$$\langle f_n, f_m\rangle = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0, & n \neq m, \\ \pi, & n = m. \end{cases}$$

$f_1, f_2, \dots$ is an orthogonal system.

For any function $F \in C[-\pi,\pi]$ consider a series

$$\frac{\langle F, f_1\rangle}{\langle f_1, f_1\rangle} f_1(x) + \frac{\langle F, f_2\rangle}{\langle f_2, f_2\rangle} f_2(x) + \frac{\langle F, f_3\rangle}{\langle f_3, f_3\rangle} f_3(x) + \cdots$$

$$\frac{\langle F, f_1\rangle}{\langle f_1, f_1\rangle} f_1(x) + \frac{\langle F, f_2\rangle}{\langle f_2, f_2\rangle} f_2(x) + \frac{\langle F, f_3\rangle}{\langle f_3, f_3\rangle} f_3(x) + \cdots$$

$$= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots,$$

where $b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} F(y) \sin(ny)\, dy.$

Theorem The above series converges to some function $G \in C[-\pi, \pi]$ with respect to the distance

$$\mathrm{dist}(f, g) = \|f - g\| = \left(\int_{-\pi}^{\pi} |f(x) - g(x)|^2\, dx \right)^{1/2}.$$

Since $\sin(-nx) = -\sin(nx)$, it follows that $G(-x) = -G(x)$.

Example. $F(x) = e^x$.

In this case, the series converges to the function $G(x) = \sinh x$. Note that $G(x) = \frac{1}{2}\big(F(x) - F(-x)\big)$.

$$h_1(x) = \cos x,\ h_2(x) = \cos 2x,\ \dots,\ h_n(x) = \cos nx,\ \dots$$

$$\langle h_n, h_m \rangle = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0, & n \neq m, \\ \pi, & n = m. \end{cases}$$

$h_1, h_2, \dots$ is an orthogonal system.

For any function $F \in C[-\pi, \pi]$ consider a series

$$\frac{\langle F, h_1 \rangle}{\langle h_1, h_1 \rangle} h_1(x) + \frac{\langle F, h_2 \rangle}{\langle h_2, h_2 \rangle} h_2(x) + \frac{\langle F, h_3 \rangle}{\langle h_3, h_3 \rangle} h_3(x) + \cdots$$

$$\frac{\langle F, h_1 \rangle}{\langle h_1, h_1 \rangle} h_1(x) + \frac{\langle F, h_2 \rangle}{\langle h_2, h_2 \rangle} h_2(x) + \frac{\langle F, h_3 \rangle}{\langle h_3, h_3 \rangle} h_3(x) + \cdots$$

$$= a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots,$$

where $a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} F(y) \cos(ny)\, dy$.

Theorem The above series converges to some function $H \in C[-\pi, \pi]$ with respect to the distance $\mathrm{dist}(f, g) = \|f - g\|$.

Since $\cos(-nx) = \cos(nx)$, it follows that $H(-x) = H(x)$. Since $\int_{-\pi}^{\pi} \cos(nx)\, dx = 0$, it follows that $\int_{-\pi}^{\pi} H(x)\, dx = 0$.

Example. $F(x) = e^x$.

In this case, the series converges to the function $H(x) = \cosh x - \pi^{-1} \sinh \pi$.

$$h_0(x) = 1,\ h_1(x) = \cos x,\ \dots,\ h_n(x) = \cos nx,\ \dots,$$
$$f_1(x) = \sin x,\ f_2(x) = \sin 2x,\ \dots,\ f_n(x) = \sin nx,\ \dots$$

This is an orthogonal system: $\langle h_0, h_0 \rangle = 2\pi$, $\langle h_n, h_n \rangle = \langle f_n, f_n \rangle = \pi$ for $n \geq 1$, while the other inner products are equal to 0.

This orthogonal system is **maximal**.

Fourier series

Definition. **Fourier series** is a series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

To each integrable function $F : [-\pi, \pi] \to \mathbb{R}$ we associate a Fourier series such that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)\, dx$$

and for $n \geq 1$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos nx \, dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin nx \, dx.$$

Convergence theorems

Theorem 1 Fourier series of a continuous function on $[-\pi, \pi]$ converges to this function with respect to the distance

$$\operatorname{dist}(f, g) = \|f - g\| = \left(\int_{-\pi}^{\pi} |f(x) - g(x)|^2 \, dx \right)^{1/2}.$$

However convergence in the sense of Theorem 1 need not imply pointwise convergence.

Theorem 2 Fourier series of a smooth function on $[-\pi, \pi]$ converges pointwise to this function on the open interval $(-\pi, \pi)$.

Example. Fourier series of the function $F(x) = x$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = -\frac{1}{n\pi}\int_{-\pi}^{\pi} x(\cos nx)'\,dx$$

$$= -\frac{1}{n\pi}\,x\cos(nx)\,\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos nx\,dx$$

$$= -\frac{1}{n\pi}\cdot 2\pi\cos(n\pi) = (-1)^{n+1}\,\frac{2}{n}.$$

Example. Fourier series of the function $F(x) = x$.

$$\boxed{2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}}$$

$$= 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \cdots \right)$$

The series converges to the function $F(x)$ for any $-\pi < x < \pi$.

For $x = \pi/2$ we obtain:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$