MATH 311-504

Topics in Applied Mathematics

Lecture 3-13: Fourier's solution of the heat equation. Review for the final exam.

Heat equation

Heat conduction in a rod is described by **one-dimensional heat equation**:

$$\boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q}$$

$K_0 = K_0(x)$, $c = c(x)$, $\rho = \rho(x)$, $Q = Q(x, t)$.

Assuming K_0, c, ρ are constant (uniform rod) and $Q = 0$ (no heat sources), we obtain

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2},$$

where $k = K_0(c\rho)^{-1}$ is called the *thermal diffusivity*.

Initial and boundary conditions

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad x_1 \le x \le x_2.$$

Initial condition: $u(x,0) = f(x),\ \ x_1 \le x \le x_2.$

Examples of boundary conditions:

- $u(x_1,t) = u_2(x_2,t) = 0.$

(constant temperature at the ends)

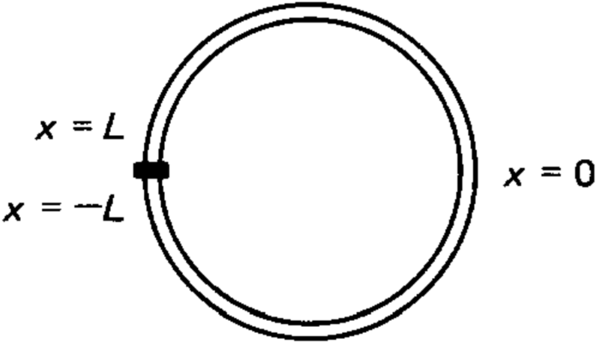
- $\frac{\partial u}{\partial x}(x_1,t) = \frac{\partial u}{\partial x}(x_2,t) = 0.$

(insulated ends)

- $u(x_1,t) = u(x_2,t),\ \ \frac{\partial u}{\partial x}(x_1,t) = \frac{\partial u}{\partial x}(x_2,t).$

(periodic boundary conditions)

Heat conduction in a thin circular ring



Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x) \quad (-\pi \le x \le \pi),$$

$$u(-\pi,t) = u(\pi,t), \quad \tfrac{\partial u}{\partial x}(-\pi,t) = \tfrac{\partial u}{\partial x}(\pi,t).$$

For any $t \ge 0$ the function $u(x,t)$ can be expanded into Fourier series:

$$u(x,t) = A_0(t) + \sum_{n=1}^{\infty} (A_n(t)\cos nx + B_n(t)\sin nx).$$

Let's assume that the series can be differentiated term-by-term. Then

$$\tfrac{\partial u}{\partial t}(x,t) = A_0'(t) + \sum_{n=1}^{\infty} (A_n'(t)\cos nx + B_n'(t)\sin nx),$$

$$\tfrac{\partial^2 u}{\partial x^2}(x,t) = \sum_{n=1}^{\infty} (-n^2)(A_n(t)\cos nx + B_n(t)\sin nx).$$

It follows that $A_0' = 0$, $A_n' = -n^2 k A_n$ and $B_n' = -n^2 k B_n$, $n \geq 1$.

Solving these ODEs, we obtain

$$A_0(t) = a_0,\ A_n(t) = a_n e^{-n^2 kt},\ B_n(t) = b_n e^{-n^2 kt},$$

where $a_i, b_j \in \mathbb{R}$. Thus

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-n^2 kt}(a_n \cos nx + b_n \sin nx).$$

Observe that a_n, b_n are Fourier coefficients of the initial data $f(x)$.

How do we solve the initial-boundary value problem?

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x) \qquad (-\pi \le x \le \pi),$$

$$u(-\pi,t) = u(\pi,t), \qquad \tfrac{\partial u}{\partial x}(-\pi,t) = \tfrac{\partial u}{\partial x}(\pi,t).$$

- Expand the function f into Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

- Write the solution:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-n^2kt}(a_n \cos nx + b_n \sin nx).$$

J. Fourier, The Analytical Theory of Heat
(written in 1807, published in 1822)

Why does it work?

Let V denote the vector space of 2π-periodic smooth functions on the real line.

Consider a linear operator $L : V \to V$ given by $L(F) = kF''$. Then the heat equation can be represented as a linear ODE on the space V:

$$\frac{dF}{dt} = L(F).$$

It turns out that functions

$$1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots$$

are eigenfunctions of the operator L.

Topics for the final exam: Part I

- n-dimensional vectors, dot product, cross product.
- Elementary analytic geometry: lines and planes.
- Systems of linear equations: elementary operations, echelon and reduced form.
- Matrix algebra, inverse matrices.
- Determinants: explicit formulas for 2-by-2 and 3-by-3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Bases and dimension.
- Linear mappings/transformations/operators.
- Subspaces. Image and null-space of a linear map.
- Matrix of a linear map relative to a basis. Change of coordinates.
- Eigenvalues and eigenvectors. Characteristic polynomial of a matrix. Bases of eigenvectors (diagonalization).

Topics for the final exam: Part III

- Norms. Inner products.
- Orthogonal and orthonormal bases. The Gram-Schmidt orthogonalization process.
- Orthogonal polynomials.
- Orthonormal bases of eigenvectors. Symmetric matrices.
- Orthogonal matrices. Rotations in space.

Problem. Let $f_1, f_2, f_3, \dots$ be the Fibonacci numbers defined by $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Find $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$.

For any integer $n \geq 1$ let $\mathbf{v}_n = (f_{n+1}, f_n)$. Then

$$\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}.$$

That is, $\mathbf{v}_{n+1} = A\mathbf{v}_n$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

In particular, $\mathbf{v}_2 = A\mathbf{v}_1$, $\mathbf{v}_3 = A\mathbf{v}_2 = A^2\mathbf{v}_1$, $\mathbf{v}_4 = A\mathbf{v}_3 = A^3\mathbf{v}_1$. In general, $\mathbf{v}_n = A^{n-1}\mathbf{v}_1$.

Characteristic equation of the matrix A:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 - \lambda - 1 = 0.$$

Eigenvalues: $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

Let $\mathbf{w}_1 = (x_1, y_1)$ and $\mathbf{w}_2 = (x_2, y_2)$ be eigenvectors of A associated with the eigenvalues λ_1 and λ_2. Then $\mathbf{w}_1, \mathbf{w}_2$ is a basis for $\mathbb{R}^2$.

In particular, $\mathbf{v}_1 = (1, 1) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2$ for some $c_1, c_2 \in \mathbb{R}$. It follows that

$$\mathbf{v}_n = A^{n-1}\mathbf{v}_1 = A^{n-1}(c_1\mathbf{w}_1 + c_2\mathbf{w}_2)$$

$$= c_1A^{n-1}\mathbf{w}_1 + c_2A^{n-1}\mathbf{w}_2 = c_1\lambda_1^{n-1}\mathbf{w}_1 + c_2\lambda_2^{n-1}\mathbf{w}_2.$$

$$\mathbf{v}_n = c_1\lambda_1^{n-1}\mathbf{w}_1 + c_2\lambda_2^{n-1}\mathbf{w}_2$$

$$\implies \quad f_n = c_1\lambda_1^{n-1}y_1 + c_2\lambda_2^{n-1}y_2.$$

Recall that $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

We have $\lambda_1 > 1$ and $-1 < \lambda_2 < 0$.

Therefore

$$\frac{f_{n+1}}{f_n} = \frac{c_1\lambda_1^n y_1 + c_2\lambda_2^n y_2}{c_1\lambda_1^{n-1}y_1 + c_2\lambda_2^{n-1}y_2}$$

$$= \lambda_1 \frac{c_1y_1 + c_2(\lambda_2/\lambda_1)^n y_2}{c_1y_1 + c_2(\lambda_2/\lambda_1)^{n-1}y_2} \to \lambda_1 \frac{c_1y_1}{c_1y_1} = \lambda_1$$

provided that $c_1y_1 \neq 0$.

Thus $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lambda_1 = \frac{1+\sqrt{5}}{2}$.