MATH 311-504

Topics in Applied Mathematics

Lecture 3-4: Norms induced by inner products. Orthogonality.

Norm

The notion of *norm* generalizes the notion of length of a vector in $\mathbb{R}^n$.

Definition. Let V be a vector space. A function $\alpha : V \to \mathbb{R}$, usually denoted $\alpha(\mathbf{x}) = \|\mathbf{x}\|$, is called a **norm** on V if it has the following properties:

(i) $\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
(ii) $\|r\mathbf{x}\| = |r|\,\|\mathbf{x}\|$ for all $r \in \mathbb{R}$ (homogeneity)
(iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

The norm defines a distance function on the vector space: $\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

Examples. $V = \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

- $\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$.
- $\|\mathbf{x}\|_p = \big(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\big)^{1/p}$, $p \geq 1$.

In particular,

- $\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$,
- $\|\mathbf{x}\|_2 = \big(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2\big)^{1/2} = |\mathbf{x}|$.

Examples. $V = C[a,b]$, $f : [a,b] \to \mathbb{R}$.

- $\|f\|_\infty = \max_{a \le x \le b} |f(x)|$ (uniform norm).

- $\|f\|_1 = \int_a^b |f(x)|\,dx$.

- $\|f\|_p = \left(\int_a^b |f(x)|^p\,dx \right)^{1/p}$, $p \ge 1$.

Inner product

The notion of *inner product* generalizes the notion of dot product of vectors in $\mathbb{R}^n$.

Definition. Let V be a vector space. A function $\beta : V \times V \to \mathbb{R}$, usually denoted $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$, is called an **inner product** on V if it is positive, symmetric, and bilinear. That is, if

(i) $\langle \mathbf{x}, \mathbf{y} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (symmetry)
(iii) $\langle r\mathbf{x}, \mathbf{y} \rangle = r\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity)
(iv) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (distributive law)

Examples. $V = \mathbb{R}^n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.
- $\langle \mathbf{x}, \mathbf{y} \rangle = d_1 x_1 y_1 + d_2 x_2 y_2 + \cdots + d_n x_n y_n$,

where $d_1, d_2, \dots, d_n > 0$.

Examples. $V = C[a, b]$.

- $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$.
- $\langle f, g \rangle = \int_a^b f(x) g(x) w(x)\, dx$,

where w is bounded, piecewise continuous, and $w > 0$ everywhere on $[a, b]$.

Cauchy-Schwarz Inequality:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}\,\sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

Corollary 1 $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}|\,|\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Equivalently, for all $x_i, y_i \in \mathbb{R}$,

$$(x_1 y_1 + \cdots + x_n y_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

Corollary 2 For any $f, g \in C[a, b]$,

$$\left(\int_a^b f(x) g(x)\, dx \right)^2 \leq \int_a^b |f(x)|^2\, dx \cdot \int_a^b |g(x)|^2\, dx.$$

Norms induced by inner products

Theorem Suppose $\langle \mathbf{x}, \mathbf{y} \rangle$ is an inner product on a vector space V. Then $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is a norm.

Proof: Positivity is obvious. Homogeneity:

$$\|r\mathbf{x}\| = \sqrt{\langle r\mathbf{x}, r\mathbf{x} \rangle} = \sqrt{r^2 \langle \mathbf{x}, \mathbf{x} \rangle} = |r| \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Triangle inequality (follows from Cauchy-Schwarz's):

$$\|\mathbf{x} + \mathbf{y}\|^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle$$
$$= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$$
$$= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2$$
$$\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \, \|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

Examples. • The length of a vector in $\mathbb{R}^n$,

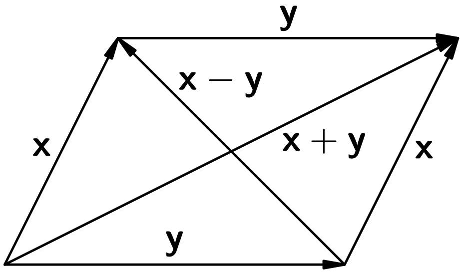
$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

is the norm induced by the dot product

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

• The norm $\|f\|_2 = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}$ on the vector space $C[a, b]$ is induced by the inner product

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$



Parallelogram Identity:

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

Proof: $\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$.

Similarly, $\|\mathbf{x}-\mathbf{y}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$.

Then $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{y}, \mathbf{y} \rangle = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$.

Example. Norms on $\mathbb{R}^n$, $n \geq 2$:

- $\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$,
- $\|\mathbf{x}\|_p = \big(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\big)^{1/p}$, $\quad p \geq 1$.

Theorem The norms $\|\mathbf{x}\|_\infty$ and $\|\mathbf{x}\|_p$, $p \neq 2$ do not satisfy the Parallelogram Identity. Hence they are not induced by any inner product on $\mathbb{R}^n$.

Hint: A counterexample to the Parallelogram Identity is provided by vectors $\mathbf{x} = (1, 0, 0, \ldots, 0)$ and $\mathbf{y} = (0, 1, 0, \ldots, 0)$.

Angle

Since $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$, we can define the *angle* between nonzero vectors in any vector space with an inner product (and induced norm):

$$\angle(\mathbf{x}, \mathbf{y}) = \arccos \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|}.$$

Then $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos \angle(\mathbf{x}, \mathbf{y})$.

In particular, vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Orthogonal systems

Let V be an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$.

Definition. A nonempty set $S \subset V$ is called an **orthogonal system** if all vectors in S are mutually orthogonal. That is, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for any $\mathbf{x}, \mathbf{y} \in S$, $\mathbf{x} \neq \mathbf{y}$.

An orthogonal system $S \subset V$ is called **orthonormal** if $\|\mathbf{x}\| = 1$ for any $\mathbf{x} \in S$.

Remark. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ form an orthonormal system if and only if

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Examples. • $V = \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.

The standard basis $\mathbf{e}_1 = (1, 0, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$, $\dots$, $\mathbf{e}_n = (0, 0, 0, \dots, 1)$.

It is an orthonormal system.

• $V = \mathbb{R}^3$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.

$\mathbf{v}_1 = (3, 5, 4)$, $\mathbf{v}_2 = (3, -5, 4)$, $\mathbf{v}_3 = (4, 0, -3)$.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0, \quad \mathbf{v}_1 \cdot \mathbf{v}_3 = 0, \quad \mathbf{v}_2 \cdot \mathbf{v}_3 = 0,$$
$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 50, \quad \mathbf{v}_2 \cdot \mathbf{v}_2 = 50, \quad \mathbf{v}_3 \cdot \mathbf{v}_3 = 25.$$

Thus the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal but not orthonormal. An orthonormal set is formed by normalized vectors $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$, $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$.

- $V = C[-\pi, \pi]$, $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$.

$f_1(x) = \sin x$, $f_2(x) = \sin 2x$, …, $f_n(x) = \sin nx$, …

$$\langle f_m, f_n \rangle = \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}\big(\cos(mx - nx) - \cos(mx + nx)\big)\,dx.$$

$$\int_{-\pi}^{\pi} \cos(kx)\,dx = \frac{\sin(kx)}{k}\Big|_{x=-\pi}^{\pi} = 0 \quad \text{if} \quad k \in \mathbb{Z},\ k \neq 0.$$

$$k = 0 \implies \int_{-\pi}^{\pi} \cos(kx)\,dx = \int_{-\pi}^{\pi} dx = 2\pi.$$

$$\langle f_m, f_n \rangle = \frac{1}{2} \int_{-\pi}^{\pi} \Big(\cos(m-n)x - \cos(m+n)x \Big) \, dx$$

$$= \begin{cases} \pi & \text{if } \; m = n \\ 0 & \text{if } \; m \neq n \end{cases}$$

Thus the set $\{f_1, f_2, f_3, \dots\}$ is orthogonal but not orthonormal.

It is orthonormal with respect to a scaled inner product

$$\langle\langle f, g \rangle\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, dx.$$

Orthogonality $\Longrightarrow$ linear independence

Theorem Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are nonzero vectors that form an orthogonal set. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Proof: Suppose $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$ for some $t_1, t_2, \ldots, t_k \in \mathbb{R}$.

Then for any index $1 \leq i \leq k$ we have

$$\langle t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k, \mathbf{v}_i\rangle = \langle \mathbf{0}, \mathbf{v}_i\rangle = 0.$$

$$\Longrightarrow \; t_1\langle \mathbf{v}_1, \mathbf{v}_i\rangle + t_2\langle \mathbf{v}_2, \mathbf{v}_i\rangle + \cdots + t_k\langle \mathbf{v}_k, \mathbf{v}_i\rangle = 0$$

By orthogonality, $t_i\langle \mathbf{v}_i, \mathbf{v}_i\rangle = 0 \Longrightarrow t_i = 0$.

Orthonormal bases

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be an orthonormal basis for an inner product space V.

Theorem Let $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$ and $\mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n$, where $x_i, y_j \in \mathbb{R}$. Then

(i) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$,

(ii) $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

Proof: (ii) follows from (i) when $\mathbf{y} = \mathbf{x}$.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \left\langle \sum_{i=1}^{n} x_i\mathbf{v}_i, \sum_{j=1}^{n} y_j\mathbf{v}_j \right\rangle = \sum_{i=1}^{n} x_i \left\langle \mathbf{v}_i, \sum_{j=1}^{n} y_j\mathbf{v}_j \right\rangle$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_iy_j\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \sum_{i=1}^{n} x_iy_i.$$