MATH 311-504

Topics in Applied Mathematics

Lecture 3-8:
Orthogonal polynomials (continued).
Symmetric matrices.

Orthogonal polynomials

$\mathcal{P}$: the vector space of all polynomials with real coefficients: $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

Suppose that $\mathcal{P}$ is endowed with an inner product.

Definition. **Orthogonal polynomials** (relative to the inner product) are polynomials $p_0, p_1, p_2, \ldots$ such that $\deg p_n = n$ (p_0 is a nonzero constant) and $\langle p_n, p_m \rangle = 0$ for $n \neq m$.

Orthogonal polynomials can be obtained by applying the Gram-Schmidt orthogonalization process to the basis $1, x, x^2, \ldots$.

Theorem (a) Orthogonal polynomials always exist.
(b) The orthogonal polynomial of a fixed degree is unique up to scaling.
(c) A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, q \rangle = 0$ for any polynomial q with $\deg q < \deg p$.
(d) A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, x^k \rangle = 0$ for any $0 \leq k < \deg p$.

Example. $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$

Orthogonal polynomials relative to this inner product are called the **Legendre polynomials**.

The standardization for the Legendre polynomials is $P_n(1) = 1$. Recurrent formula:

$$(n+1)P_{n+1} = (2n+1)xP_n(x) - nP_{n-1}(x).$$

$P_0(x) = 1,\ P_1(x) = x,$
$P_2(x) = \frac{1}{2}(3xP_1(x) - P_0(x)) = \frac{1}{2}(3x^2 - 1),$
$P_3(x) = \frac{1}{3}(5xP_2(x) - 2P_1(x)) = \frac{1}{2}(5x^3 - 3x),$
$P_4(x) = \frac{1}{4}(7xP_3(x) - 3P_2(x)) = \frac{1}{8}(35x^4 - 30x^2 + 3).$

Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

The best least squares fit is a polynomial $p(x)$ that minimizes the distance relative to the integral norm

$$\|f - p\| = \left(\int_{-1}^{1} |f(x) - p(x)|^2 \, dx\right)^{1/2}$$

over all polynomials of degree 2.

The norm $\|f - p\|$ is minimal if p is the orthogonal projection of the function f on the subspace $\mathcal{P}_2$ of polynomials of degree at most 2.

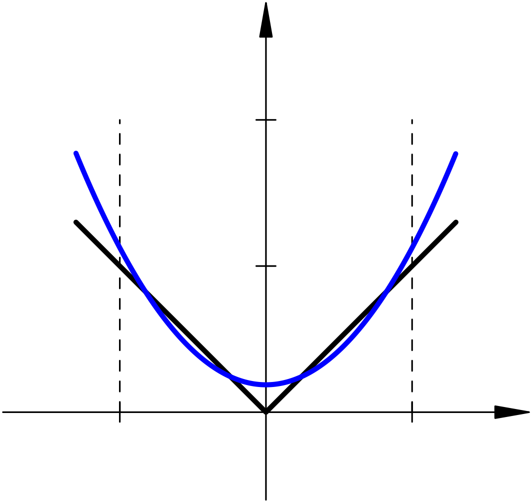
Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

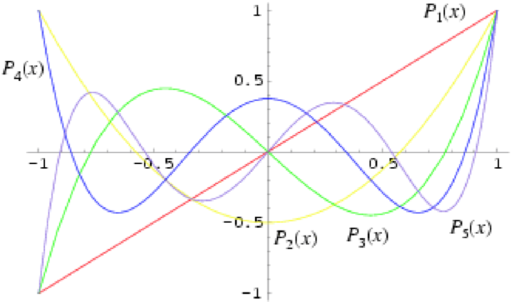
Solution:

$$p(x) = \frac{\langle f, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) + \frac{\langle f, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1(x) + \frac{\langle f, P_2 \rangle}{\langle P_2, P_2 \rangle} P_2(x)$$

$$= \frac{1}{2} P_0(x) + \frac{5}{8} P_2(x)$$

$$= \frac{1}{2} + \frac{5}{16}(3x^2 - 1) = \frac{3}{16}(5x^2 + 1).$$





Legendre polynomials

Definition. **Chebyshev polynomials** $T_0, T_1, T_2, \ldots$ are orthogonal polynomials relative to the inner product

$$\langle p, q \rangle = \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1-x^2}}\, dx,$$

with the standardization $T_n(1) = 1$.

Remark. "T" is like in "Tschebyscheff".

Change of variable in the integral: $x = \cos\phi$.

$$\langle p, q \rangle = -\int_0^{\pi} \frac{p(\cos\phi)\, q(\cos\phi)}{\sqrt{1-\cos^2\phi}} \cos'\phi\, d\phi$$
$$= \int_0^{\pi} p(\cos\phi)\, q(\cos\phi)\, d\phi.$$

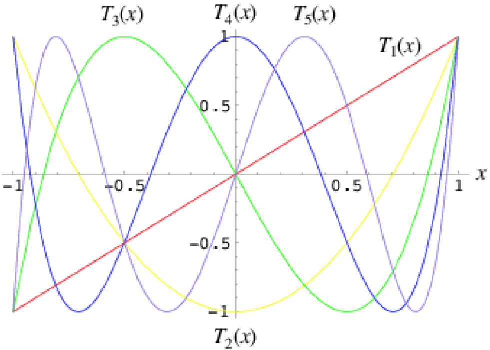
Theorem. $T_n(\cos\phi) = \cos n\phi$.

$$\langle T_n, T_m \rangle = \int_0^{\pi} T_n(\cos\phi)\, T_m(\cos\phi)\, d\phi$$
$$= \int_0^{\pi} \cos(n\phi)\cos(m\phi)\, d\phi = 0 \ \text{ if } \ n \neq m.$$

Recurrent formula: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

$T_0(x) = 1, \quad T_1(x) = x,$
$T_2(x) = 2x^2 - 1,$
$T_3(x) = 4x^3 - 3x,$
$T_4(x) = 8x^4 - 8x^2 + 1, \ \ldots$

That is, $\cos 2\phi = 2\cos^2\phi - 1,$
$\cos 3\phi = 4\cos^3\phi - 3\cos\phi,$
$\cos 4\phi = 8\cos^4\phi - 8\cos^2\phi + 1, \ \ldots$



Chebyshev polynomials

Symmetric matrices

Proposition For any $n\times n$ matrix A and any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\boxed{A\mathbf{x}\cdot\mathbf{y} = \mathbf{x}\cdot A^T\mathbf{y}.}$

Proof: $A\mathbf{x}\cdot\mathbf{y} = \mathbf{y}^T A\mathbf{x} = (\mathbf{y}^T A\mathbf{x})^T = \mathbf{x}^T A^T\mathbf{y} =$
$= A^T\mathbf{y}\cdot\mathbf{x} = \mathbf{x}\cdot A^T\mathbf{y}.$

Definition. An $n\times n$ matrix A is called

- **symmetric** if $A^T = A$;
- **orthogonal** if $AA^T = A^TA = I$, that is, if $A^T = A^{-1}$;
- **normal** if $AA^T = A^TA$.

Clearly, symmetric and orthogonal matrices are normal.

Theorem If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of a symmetric matrix A associated with different eigenvalues, then $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof: Suppose $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$, where $\lambda \neq \mu$. Then $A\mathbf{x} \cdot \mathbf{y} = \lambda(\mathbf{x} \cdot \mathbf{y})$, $\mathbf{x} \cdot A\mathbf{y} = \mu(\mathbf{x} \cdot \mathbf{y})$.
But $A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^T\mathbf{y} = \mathbf{x} \cdot A\mathbf{y}$.
Thus $\lambda(\mathbf{x} \cdot \mathbf{y}) = \mu(\mathbf{x} \cdot \mathbf{y}) \implies \mathbf{x} \cdot \mathbf{y} = 0$.

Theorem Suppose A is a symmetric $n \times n$ matrix. Then **(a)** all eigenvalues of A are real;
(b) there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Example. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

- A is symmetric.
- A has three eigenvalues: 0, 2, and 3.
- Associated eigenvectors are $\mathbf{v}_1 = (-1, 0, 1)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (0, 1, 0)$.
- Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthogonal basis for $\mathbb{R}^3$.