Math 311-504

Topics in Applied Mathematics

Lecture 7:
Linear independence (continued).
Matrix algebra.

Linear independence

Definition. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are called **linearly dependent** if they satisfy a relation

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $t_1, \dots, t_k \in \mathbb{R}$ are not all equal to zero. Otherwise the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0} \implies t_1 = \cdots = t_k = 0.$$

Theorem The vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent if and only if one of them is a linear combination of the others.

Definition. A subset $S \subset \mathbb{R}^n$ is called a **hyperplane** (or an **affine subspace**) if it has a parametric representation $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k + \mathbf{v}_0$, where $\mathbf{v}_i$ are fixed n-dimensional vectors and t_i are arbitrary scalars.

The number k of parameters may depend on a representation. The hyperplane S is called a k-**plane** if k is as small as possible.

Theorem A hyperplane

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k + \mathbf{v}_0$$

is a k-plane if and only if vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Examples

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ in $\mathbb{R}^3$.

$t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3 = \mathbf{0} \implies (t_1, t_2, t_3) = \mathbf{0}$
$\implies t_1 = t_2 = t_3 = 0$

Thus $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent.

- Vectors $\mathbf{v}_1 = (4, 3, 0, 1)$, $\mathbf{v}_2 = (1, -1, 2, 0)$, and $\mathbf{v}_3 = (-2, 2, -4, 0)$ in $\mathbb{R}^4$.

It is easy to observe that $\mathbf{v}_3 = -2\mathbf{v}_2$.
$\implies 0\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = \mathbf{0}$

Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent. At the same time, the vector $\mathbf{v}_1$ is not a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

- Vectors $\mathbf{u}_1 = (1, 2, 0)$, $\mathbf{u}_2 = (3, 1, 1)$, and $\mathbf{u}_3 = (4, -7, 3)$ in $\mathbb{R}^3$.

We need to check if the vector equation $t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + t_3\mathbf{u}_3 = \mathbf{0}$ has solutions other than $t_1 = t_2 = t_3 = 0$.

This vector equation is equivalent to a system

$$\begin{cases} r_1 + 3r_2 + 4r_3 = 0, \\ 2r_1 + r_2 - 7r_3 = 0, \\ r_2 + 3r_3 = 0. \end{cases} \qquad \left(\begin{array}{ccr|c} 1 & 3 & 4 & 0 \\ 2 & 1 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right)$$

Row reduction yields:

$$\left(\begin{array}{ccr|c} 1 & 3 & 4 & 0 \\ 2 & 1 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{crr|c} 1 & 3 & 4 & 0 \\ 0 & -5 & -15 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \color{red}{1} & 3 & 4 & 0 \\ 0 & \color{red}{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The variable t_3 is free $\implies$ there are infinitely many solutions $\implies$ the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent.

Matrices

Definition. An **m-by-n matrix** is a rectangular array of numbers that has m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Notation: $A = (a_{ij})_{1 \le i \le n,\, 1 \le j \le m}$ or simply $A = (a_{ij})$ if the dimensions are known.

An n-dimensional vector can be represented as a $1 \times n$ matrix (row vector) or as an $n \times 1$ matrix (column vector):

$$(x_1, x_2, \ldots, x_n) \qquad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

An $m \times n$ matrix $A = (a_{ij})$ can be regarded as a column of n-dimensional row vectors or as a row of m-dimensional column vectors:

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}, \qquad \mathbf{v}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$$

$$A = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n), \qquad \mathbf{w}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Vector algebra

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be n-dimensional vectors, and $r \in \mathbb{R}$ be a scalar.

Vector sum: $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

Scalar multiple: $r\mathbf{a} = (ra_1, ra_2, \ldots, ra_n)$

Zero vector: $\mathbf{0} = (0, 0, \ldots, 0)$

Negative of a vector: $-\mathbf{b} = (-b_1, -b_2, \ldots, -b_n)$

Vector difference:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$$

Matrix algebra

Definition. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. The **sum** $A + B$ is defined to be the $m \times n$ matrix $C = (c_{ij})$ such that $c_{ij} = a_{ij} + b_{ij}$ for all indices i, j.

That is, two matrices with the same dimensions can be added by adding their corresponding entries.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{pmatrix}$$

Definition. Given an $m \times n$ matrix $A = (a_{ij})$ and a number r, the **scalar multiple** rA is defined to be the $m \times n$ matrix $D = (d_{ij})$ such that $d_{ij} = ra_{ij}$ for all indices i, j.

That is, to multiply a matrix by a scalar r, one multiplies each entry of the matrix by r.

$$r \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{pmatrix}$$

The $m \times n$ **zero matrix** (all entries are zeros) is denoted O_{mn} or simply O.

Negative of a matrix: $-A$ is defined as $(-1)A$.

Matrix **difference**: $A - B$ is defined as $A + (-B)$.

As far as the *linear operations* (addition and scalar multiplication) are concerned, the $m \times n$ matrices can be regarded as mn-dimensional vectors.

Examples

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} 5 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \qquad A - B = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \end{pmatrix},$$

$$2C = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \qquad 3D = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix},$$

$$2C + 3D = \begin{pmatrix} 7 & 3 \\ 0 & 5 \end{pmatrix}, \qquad A + D \text{ is not defined.}$$

Properties of linear operations

$$(A + B) + C = A + (B + C)$$
$$A + B = B + A$$
$$A + O = O + A = A$$
$$A + (-A) = (-A) + A = O$$
$$r(sA) = (rs)A$$
$$r(A + B) = rA + rB$$
$$(r + s)A = rA + sA$$
$$1A = A$$
$$0A = O$$

Dot product

Definition. The **dot product** of n-dimensional vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is a scalar

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{k=1}^{n} x_k y_k.$$

Matrix multiplication

The product of matrices A and B is defined if the number of columns in A matches the number of rows in B.

Definition. Let $A = (a_{ik})$ be an $m \times n$ matrix and $B = (b_{kj})$ be an $n \times p$ matrix. The **product** AB is defined to be the $m \times p$ matrix $C = (c_{ij})$ such that $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ for all indices i, j.

That is, matrices are multiplied **row by column**:

$$\begin{pmatrix} * & * & * \\ \boxed{*} & \boxed{*} & \boxed{*} \end{pmatrix} \begin{pmatrix} * & * & \boxed{*} & * \\ * & * & \boxed{*} & * \\ * & * & \boxed{*} & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & \boxed{*} & * \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \hline a_{21} & a_{22} & \dots & a_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

$$B = \left(\begin{array}{c|c|c|c} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{array}\right) = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p)$$

$$\implies AB = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{w}_1 & \mathbf{v}_1 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_1 \cdot \mathbf{w}_p \\ \mathbf{v}_2 \cdot \mathbf{w}_1 & \mathbf{v}_2 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_2 \cdot \mathbf{w}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_m \cdot \mathbf{w}_1 & \mathbf{v}_m \cdot \mathbf{w}_2 & \dots & \mathbf{v}_m \cdot \mathbf{w}_p \end{pmatrix}$$

Examples.

$$(x_1, x_2, \ldots, x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = (\textstyle\sum_{k=1}^n x_k y_k),$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} (x_1, x_2, \ldots, x_n) = \begin{pmatrix} y_1 x_1 & y_1 x_2 & \ldots & y_1 x_n \\ y_2 x_1 & y_2 x_2 & \ldots & y_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ y_n x_1 & y_n x_2 & \ldots & y_n x_n \end{pmatrix}.$$

Example.

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & 17 & 16 & 1 \end{pmatrix}$$

Any system of linear equations can be rewritten as a matrix equation.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\iff \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Properties of matrix multiplication:

$(AB)C = A(BC)$ (associative law)

$(A + B)C = AC + BC$ (distributive law #1)

$C(A + B) = CA + CB$ (distributive law #2)

$(rA)B = A(rB) = r(AB)$

(Any of the above identities holds provided that matrix sums and products are well defined.)