MATH 311–504 March 28, 2008

Test 2

Problem 1 (20 pts.) Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x - y + 2z = 0$.
(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + 2y + 3z = 6$.
(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y = z^2$.
(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 = 0$.

Problem 2 (20 pts.) Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the space of 2-by-2 matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the matrix of the operator L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 3 (30 pts.) Consider a linear operator $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -3 \\ 2 & 1 & 4 \end{pmatrix}.$$

(i) Find a basis for the image of f.
(ii) Find a basis for the null-space of f.

Problem 4 (30 pts.) Let $B = \begin{pmatrix} -1 & 1 \\ 5 & 3 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.
(ii) For each eigenvalue of B, find an associated eigenvector.
(iii) Is there a basis for $\mathbb{R}^2$ consisting of eigenvectors of B? Explain.
(iv) Find all eigenvalues of the matrix B^2.

Bonus Problem 5 (20 pts.) Solve the following system of differential equations (find all solutions):

$$\begin{cases} \dfrac{dx}{dt} = -x + y, \\ \dfrac{dy}{dt} = 5x + 3y. \end{cases}$$