MATH 311–504 Spring 2008

Sample problems for Test 2: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (20 pts.) Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x,y,z)\in\mathbb{R}^3$ such that $xyz=0$.
(ii) The set S_2 of vectors $(x,y,z)\in\mathbb{R}^3$ such that $x+y+z=0$.
(iii) The set S_3 of vectors $(x,y,z)\in\mathbb{R}^3$ such that $y^2+z^2=0$.
(iv) The set S_4 of vectors $(x,y,z)\in\mathbb{R}^3$ such that $y^2-z^2=0$.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, a subspace must not be empty.

It is easy to see that each of the sets S_1, S_2, S_3, and S_4 contains the zero vector $(0,0,0)$ and all these sets are closed under scalar multiplication.

The set S_1 is the union of three planes $x=0$, $y=0$, and $z=0$. It is not closed under addition as the following example shows: $(1,1,0)+(0,0,1)=(1,1,1)$.

S_2 is a plane passing through the origin. Obviously, it is closed under addition.

The condition $y^2+z^2=0$ is equivalent to $y=z=0$. Hence S_3 is a line passing through the origin. It is closed under addition.

Since $y^2-z^2=(y-z)(y+z)$, the set S_4 is the union of two planes $y-z=0$ and $y+z=0$. The following example shows that S_4 is not closed under addition: $(0,1,1)+(0,1,-1)=(0,2,0)$.

Thus S_2 and S_3 are subspaces of $\mathbb{R}^3$ while S_1 and S_4 are not.

Problem 2 (20 pts.) Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the space of 2-by-2 matrices with real entries. Consider a linear operator $L:\mathcal{M}_{2,2}(\mathbb{R})\to\mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Find the matrix of the operator L with respect to the basis

$$E_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad E_2=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad E_3=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad E_4=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let M_L denote the desired matrix. By definition, M_L is a 4-by-4 matrix whose columns are coordinates of the matrices $L(E_1), L(E_2), L(E_3), L(E_4)$ with respect to the basis E_1, E_2, E_3, E_4. We have that

$$L(E_1)=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}=1E_1+0E_2+3E_3+0E_4,$$

$$L(E_2)=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}=0E_1+1E_2+0E_3+3E_4,$$

$$L(E_3)=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}=2E_1+0E_2+4E_3+0E_4,$$

$$L(E_4)=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}=0E_1+2E_2+0E_3+4E_4.$$

It follows that

$$M_L = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}.$$

Problem 3 (30 pts.) Consider a linear operator $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}.$$

(i) Find a basis for the image of f.

The image of the linear operator f is the subspace of $\mathbb{R}^3$ spanned by columns of the matrix A, that is, by vectors $\mathbf{v}_1 = (1, -2, -1)$, $\mathbf{v}_2 = (-1, 1, 0)$, and $\mathbf{v}_3 = (-2, 3, 1)$. The third column is a linear combination of the first two, $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$. Therefore the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is the same as the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent because they are not parallel. Thus $\mathbf{v}_1$, $\mathbf{v}_2$ is a basis for the image of f.

Alternative solution: The image of f is spanned by columns of the matrix A, that is, by vectors $\mathbf{v}_1 = (1, -2, -1)$, $\mathbf{v}_2 = (-1, 1, 0)$, and $\mathbf{v}_3 = (-2, 3, 1)$. To check linear independence of these vectors, we evaluate the determinant of A (using expansion by the third row):

$$\det A = \begin{vmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} -1 & -2 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} = (-1) \cdot (-1) + 1 \cdot (-1) = 0.$$

Since $\det A = 0$, the columns of the matrix A are linearly dependent. Then the image of f is at most two-dimensional. On the other hand, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent because they are not parallel. Hence they span a two-dimensional subspace of $\mathbb{R}^3$. It follows that this subspace coincides with the image of f. Therefore $\mathbf{v}_1$, $\mathbf{v}_2$ is a basis for the image of f.

(ii) Find a basis for the null-space of f.

The null-space of f is the set of solutions of the vector equation $A\mathbf{x} = \mathbf{0}$. To solve the equation, we shall convert the matrix A to reduced row echelon form. Since the right-hand side of the equation is the zero vector, elementary row operations do not change the solution set.

First we add the first row of the matrix A twice to the second row and once to the third one:

$$\begin{pmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Then we subtract the second row from the third row:

$$\begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, we multiply the second row by -1 and add it to the first row:

$$\begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that the vector equation $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $x - z = y + z = 0$, where $\mathbf{x} = (x, y, z)$. The general solution of the system is $x = t$, $y = -t$, $z = t$ for an arbitrary $t \in \mathbb{R}$. That is, $\mathbf{x} = (t, -t, t) = t(1, -1, 1)$, where $t \in \mathbb{R}$. Thus the null-space of the linear operator f is the line $t(1, -1, 1)$. The vector $(1, -1, 1)$ is a basis for this line.

Problem 4 (30 pts.) Let $B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. We obtain that

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 2(1-\lambda) - 2(1-\lambda)$$

$$= (1-\lambda)\big((1-\lambda)^2 - 4\big) = (1-\lambda)\big((1-\lambda) - 2\big)\big((1-\lambda) + 2\big) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix B has three eigenvalues: -1, 1, and 3.

(ii) For each eigenvalue of B, find an associated eigenvector.

An eigenvector $\mathbf{v} = (x, y, z)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{v} = \mathbf{0}$. To solve the equation, we apply row reduction to the matrix $B - \lambda I$.

First consider the case $\lambda = -1$. The row reduction yields

$$B + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(B + I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)$ is an eigenvector of B associated with the eigenvalue -1.

Secondly, consider the case $\lambda = 1$. The row reduction yields

$$B - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(B - I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, 0, 1)$ is an eigenvector of B associated with the eigenvalue 1.

Finally, consider the case $\lambda = 3$. The row reduction yields

$$B - 3I = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(B - 3I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = t$, $y = t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of B associated with the eigenvalue 3.

(iii) Is there a basis for $\mathbb{R}^3$ consisting of eigenvectors of B? Explain.

By the above the vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B. These vectors are linearly independent since

$$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -2 \cdot 2 = -4 \neq 0.$$

It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

Alternatively, the existence of a basis for $\mathbb{R}^3$ consisting of eigenvectors of B already follows from the fact that the matrix B has three distinct eigenvalues.

(iv) Find a diagonal matrix D and an invertible matrix U such that $B = UDU^{-1}$.

We have that $B = UDU^{-1}$, where

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

This follows from the fact that D is the matrix of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{x}) = B\mathbf{x}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ while U is the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis.

(v) Find all eigenvalues of the matrix B^2.

Suppose that $B\mathbf{v} = \lambda\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then

$$B^2\mathbf{v} = B(B\mathbf{v}) = B(\lambda\mathbf{v}) = \lambda(B\mathbf{v}) = \lambda^2\mathbf{v}.$$

It follows that the vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B^2 associated with eigenvalues 1, 1, and 9, respectively. Since a 3-by-3 matrix can have 3 eigenvalues, we need additional arguments to show that 1 and 9 are the only eigenvalues of B^2.

Assume that $\mathbf{v}$ is an eigenvector of B^2 associated with an eigenvalue μ. Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$, we have $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3$ for some $r_1, r_2, r_3 \in \mathbb{R}^3$. Then

$$B^2\mathbf{v} = r_1(B^2\mathbf{v}_1) + r_2(B^2\mathbf{v}_2) + r_3(B^2\mathbf{v}_3) = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + 9r_3\mathbf{v}_3, \qquad \mu\mathbf{v} = \mu r_1\mathbf{v}_1 + \mu r_2\mathbf{v}_2 + \mu r_3\mathbf{v}_3.$$

The equality $B^2\mathbf{v} = \mu\mathbf{v}$ implies that $r_1 = \mu r_1$, $r_2 = \mu r_2$, and $9r_3 = \mu r_3$. Equivalently, $(\mu-1)r_1 = (\mu-1)r_2 = (\mu-9)r_3 = 0$. As the coefficients r_1, r_2, r_3 are not all equal to 0, it follows that $\mu = 1$ or $\mu = 9$.

Bonus Problem 5 (20 pts.) Solve the following system of differential equations (find all solutions):

$$\begin{cases} \dfrac{dx}{dt} = x + 2y, \\ \dfrac{dy}{dt} = x + y + z, \\ \dfrac{dz}{dt} = 2y + z. \end{cases}$$

Introducing a vector function $\mathbf{v}(t) = (x(t), y(t), z(t))$, we can rewrite the system in the following way:

$$\frac{d\mathbf{v}}{dt} = B\mathbf{v}, \quad \text{where} \quad B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

As shown in the solution of Problem 4, there is a basis for $\mathbb{R}^3$ consisting of eigenvectors of the matrix B. Namely, $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of B associated with the eigenvalues -1, 1, and 3, respectively. These vectors form a basis for $\mathbb{R}^3$. It follows that

$$\mathbf{v}(t) = r_1(t)\mathbf{v}_1 + r_2(t)\mathbf{v}_2 + r_3(t)\mathbf{v}_3,$$

where r_1, r_2, r_3 are well-defined scalar functions (coordinates of $\mathbf{v}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$). Then

$$\frac{d\mathbf{v}}{dt} = \frac{dr_1}{dt}\mathbf{v}_1 + \frac{dr_2}{dt}\mathbf{v}_2 + \frac{dr_3}{dt}\mathbf{v}_3, \qquad B\mathbf{v} = r_1 B\mathbf{v}_1 + r_2 B\mathbf{v}_2 + r_3 B\mathbf{v}_3 = -r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + 3r_3\mathbf{v}_3.$$

As a consequence,

$$\frac{d\mathbf{v}}{dt} = B\mathbf{v} \quad \Longleftrightarrow \quad \begin{cases} \frac{dr_1}{dt} = -r_1, \\ \frac{dr_2}{dt} = r_2, \\ \frac{dr_3}{dt} = 3r_3. \end{cases}$$

The general solution of the differential equation $r_1' = -r_1$ is $r_1(t) = c_1 e^{-t}$, where c_1 is an arbitrary constant. The general solution of the equation $r_2' = r_2$ is $r_2(t) = c_2 e^t$, where c_2 is another arbitrary constant. The general solution of the equation $r_3' = 3r_3$ is $r_3(t) = c_3 e^{3t}$, where c_3 is yet another arbitrary constant. Therefore the general solution of the equation $\mathbf{v}' = B\mathbf{v}$ is

$$\mathbf{v}(t) = c_1e^{-t}\mathbf{v}_1 + c_2e^t\mathbf{v}_2 + c_3e^{3t}\mathbf{v}_3 = c_1e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2e^t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3e^{3t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1e^{-t} - c_2e^t + c_3e^{3t} \\ -c_1e^{-t} + c_3e^{3t} \\ c_1e^{-t} + c_2e^t + c_3e^{3t} \end{pmatrix},$$

where $c_1, c_2, c_3 \in \mathbb{R}$. Equivalently,

$$\begin{cases} x(t) = c_1e^{-t} - c_2e^t + c_3e^{3t}, \\ y(t) = -c_1e^{-t} + c_3e^{3t}, \\ z(t) = c_1e^{-t} + c_2e^t + c_3e^{3t}. \end{cases}$$