MATH 311

Topics in Applied Mathematics I

Lecture 14e:
Additional review for Test 1.

Vector space of infinite sequences

- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, x_3, \ldots)$, $x_n \in \mathbb{R}$

To add two infinite sequences

$$\mathbf{x} = (x_1, x_2, x_3, \ldots) \text{ and } \mathbf{y} = (y_1, y_2, y_3, \ldots),$$

we add their corresponding terms:

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3, \ldots).$$

To multiply a sequence $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ by a scalar $r \in \mathbb{R}$, we multiply each term by that scalar: $r\mathbf{x} = (rx_1,\ rx_2,\ rx_3, \ldots)$.

The zero vector in this vector space is the sequence of all zeros: $\mathbf{0} = (0, 0, 0, \ldots)$. To get the negative of a sequence $\mathbf{x} = (x_1, x_2, x_3, \ldots)$, we negate each term: $-\mathbf{x} = (-x_1, -x_2, -x_3, \ldots)$.

Problem. Determine which of the following subsets of $\mathbb{R}^\infty$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^\infty$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(i) S_1: sequences with infinitely many zero terms.

$\mathbf{0} = (0, 0, 0, \dots) \in S_1 \implies S_1$ is not empty.

Suppose $\mathbf{x} = (x_1, x_2, x_3, \dots)$ has infinitely many zero terms. Note that $x_n = 0 \implies rx_n = 0$ for all $r \in \mathbb{R}$. Therefore any scalar multiple $r\mathbf{x}$ also has infinitely many zero terms. Hence S_1 is closed under scalar multiplication.

However S_1 is not closed under addition. Counterexample: $(1, 0, 1, 0, 1, 0, \dots) + (0, 1, 0, 1, 0, 1, \dots) = (1, 1, 1, 1, 1, 1, \dots)$.

Thus S_1 is not a subspace of $\mathbb{R}^\infty$.

Problem. Determine which of the following subsets of $\mathbb{R}^\infty$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^\infty$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(ii) S_2: sequences with nonnegative terms.

$\mathbf{0} = (0, 0, 0, \dots) \in S_2 \implies S_2$ is not empty.

Suppose $\mathbf{x} = (x_1, x_2, x_3, \dots)$ and $\mathbf{y} = (y_1, y_2, y_3, \dots)$ have nonnegative terms. Then $x_n + y_n \geq 0 + 0 = 0$ for all n. Also, $rx_n \geq 0$ if $r \geq 0$. Hence $\mathbf{x} + \mathbf{y} \in S_2$ and $r\mathbf{x} \in S_2$ if $r \geq 0$. That is, the set S_2 is closed under addition and under multiplication by nonnegative scalars.

However S_2 is not closed under multiplication by negative scalars. Counterexample:

$$(-1)(1, 1, 1, 1 \dots) = (-1, -1, -1, -1 \dots).$$

Thus S_2 is not a subspace of $\mathbb{R}^\infty$.

Problem. Determine which of the following subsets of $\mathbb{R}^\infty$ are subspaces. Briefly explain.

(iii) S_3: arithmetic progressions.

A sequence $\mathbf{x} = (x_1, x_2, x_3, \dots)$ is an arithmetic progression if $x_{n+1} = x_n + d$ for some $d \in \mathbb{R}$ and all n.

$\mathbf{0} = (0, 0, 0, \dots)$ is an arithmetic progression with common difference $d = 0$. Hence $\mathbf{0} \in S_3 \implies S_3$ is not empty.

Suppose $\mathbf{x} = (x_1, x_2, x_3, \dots)$ and $\mathbf{y} = (y_1, y_2, y_3, \dots)$ are arithmetic progressions. That is, $x_{n+1} = x_n + d$ and $y_{n+1} = y_n + d'$ for some $d, d' \in \mathbb{R}$ and all n. Then $x_{n+1} + y_{n+1} = (x_n + d) + (y_n + d') = (x_n + y_n) + (d + d')$ for all n so that $\mathbf{x} + \mathbf{y}$ is an arithmetic progression with common difference $d + d'$. Also, $rx_{n+1} = rx_n + rd$ for any scalar r and all n. Hence $r\mathbf{x}$ is an arithmetic progression with common difference rd.

Therefore the set S_3 is closed under addition and scalar multiplication. Thus S_3 is a subspace of $\mathbb{R}^\infty$.

Problem. Determine which of the following subsets of $\mathbb{R}^\infty$ are subspaces. Briefly explain.

(iv) S_4: geometric progressions.

A sequence $\mathbf{x} = (x_1, x_2, x_3, \dots)$ is a geometric progression if $x_{n+1} = x_n q$ for some $q \neq 0$ and all n.

$\mathbf{0} = (0, 0, 0, \dots)$ is a geometric progression with common ratio $q = 1$. Hence $\mathbf{0} \in S_4 \implies S_4$ is not empty.

Suppose $\mathbf{x} = (x_1, x_2, x_3, \dots)$ is a geometric progression with common ratio q. Then $rx_{n+1} = r(x_n q) = (rx_n)q$ for any scalar r and all n. Hence $r\mathbf{x}$ is also a geometric progression with the same common ratio q. Therefore the set S_4 is closed under scalar multiplication.

However S_4 is not closed under addition. Counterexample: $(1, 1, 1, \dots) + (2, 4, 8, \dots, 2^n, \dots) = (3, 5, 9, \dots, 2^n{+}1, \dots)$.

Thus S_4 is not a subspace of $\mathbb{R}^\infty$.

Problem. Determine which of the following subsets of $\mathbb{R}^\infty$ are subspaces. Briefly explain.

(v) S_5: sequences of bounded variation.

A sequence $\mathbf{x} = (x_1, x_2, x_3, \dots)$ is said to have bounded variation if the series $\sum_{n=1}^{\infty} |x_{n+1} - x_n|$ converges.

$\mathbf{0} = (0, 0, 0, \dots)$ has variation $\sum_{n=1}^{\infty} |0 - 0| = 0 < \infty$.
Hence $\mathbf{0} \in S_5 \implies S_5$ is not empty.

Suppose $\mathbf{x} = (x_1, x_2, x_3, \dots)$ and $\mathbf{y} = (y_1, y_2, y_3, \dots)$ both have bounded variation. Since

$$|(x_{n+1}+y_{n+1}) - (x_n+y_n)| \le |x_{n+1} - x_n| + |y_{n+1} - y_n|$$

for all n, we obtain $\sum_{n=1}^{\infty} |(x_{n+1}+y_{n+1}) - (x_n+y_n)| \le \sum_{n=1}^{\infty} |x_{n+1}-x_n| + \sum_{n=1}^{\infty} |y_{n+1}-y_n| < \infty$. Hence $\mathbf{x} + \mathbf{y} \in S_5$.
Also, $\sum_{n=1}^{\infty} |rx_{n+1} - rx_n| = |r| \sum_{n=1}^{\infty} |x_{n+1} - x_n| < \infty$ for any scalar r so that $r\mathbf{x} \in S_5$.

Therefore the set S_5 is closed under addition and scalar multiplication. Thus S_5 is a subspace of $\mathbb{R}^\infty$.