## MATH 311 Topics in Applied Mathematics I

Lecture 20:

Matrix transformations.

Matrix of a linear transformation.

## **Matrix transformations**

Any  $m \times n$  matrix A gives rise to a transformation  $L: \mathbb{R}^n \to \mathbb{R}^m$  given by  $L(\mathbf{x}) = A\mathbf{x}$ , where  $\mathbf{x} \in \mathbb{R}^n$  and  $L(\mathbf{x}) \in \mathbb{R}^m$  are regarded as column vectors. This transformation is **linear**.

Example. 
$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
.

Let  $\mathbf{e}_1 = (1,0,0)$ ,  $\mathbf{e}_2 = (0,1,0)$ ,  $\mathbf{e}_3 = (0,0,1)$  be the standard basis for  $\mathbb{R}^3$ . We have that  $L(\mathbf{e}_1) = (1,3,0)$ ,  $L(\mathbf{e}_2) = (0,4,5)$ ,  $L(\mathbf{e}_3) = (2,7,8)$ . Thus  $L(\mathbf{e}_1)$ ,  $L(\mathbf{e}_2)$ ,  $L(\mathbf{e}_3)$  are columns of the matrix.

**Problem.** Find a linear mapping  $L: \mathbb{R}^3 \to \mathbb{R}^2$  such that  $L(\mathbf{e}_1) = (1,1)$ ,  $L(\mathbf{e}_2) = (0,-2)$ ,  $L(\mathbf{e}_3) = (3,0)$ , where  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  is the standard basis for  $\mathbb{R}^3$ .

$$= xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3)$$

$$= x(1,1) + y(0,-2) + z(3,0) = (x+3z, x-2y)$$

$$L(x,y,z) = \begin{pmatrix} x+3z \\ x-2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

 $L(x, y, z) = L(xe_1 + ye_2 + ze_3)$ 

Columns of the matrix are vectors  $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$ .

**Theorem** Suppose  $L: \mathbb{R}^n \to \mathbb{R}^m$  is a linear map. Then there exists an  $m \times n$  matrix A such that  $L(\mathbf{x}) = A\mathbf{x}$  for all  $\mathbf{x} \in \mathbb{R}^n$ . Columns of A are vectors  $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$ , where  $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$  is the standard basis for  $\mathbb{R}^n$ .

$$\mathbf{y} = A\mathbf{x} \iff \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

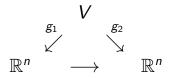
$$\iff \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

## Change of coordinates (revisited)

Let V be a vector space.

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be a basis for V and  $g_1 : V \to \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.

Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  be another basis for V and  $g_2 : V \to \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.



The composition  $g_2 \circ g_1^{-1}$  is a linear mapping of  $\mathbb{R}^n$  to itself. Hence it's represented as  $\mathbf{x} \mapsto U\mathbf{x}$ , where U is an  $n \times n$  matrix.

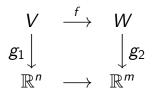
U is called the **transition matrix** from  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  to  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ . Columns of U are coordinates of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  with respect to the basis  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ .

## Matrix of a linear transformation

Let V, W be vector spaces and  $f: V \to W$  be a linear map.

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be a basis for V and  $g_1 : V \to \mathbb{R}^n$  be the coordinate mapping corresponding to this basis.

Let  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$  be a basis for W and  $g_2 : W \to \mathbb{R}^m$  be the coordinate mapping corresponding to this basis.



The composition  $g_2 \circ f \circ g_1^{-1}$  is a linear mapping of  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . Hence it's represented as  $\mathbf{x} \mapsto A\mathbf{x}$ , where A is an  $m \times n$  matrix.

A is called the **matrix of** f with respect to bases  $\mathbf{v}_1, \ldots, \mathbf{v}_n$  and  $\mathbf{w}_1, \ldots, \mathbf{w}_m$ . Columns of A are coordinates of vectors  $f(\mathbf{v}_1), \ldots, f(\mathbf{v}_n)$  with respect to the basis  $\mathbf{w}_1, \ldots, \mathbf{w}_m$ .

Examples. •  $D: \mathcal{P}_3 \to \mathcal{P}_2$ , (Dp)(x) = p'(x). Let  $A_D$  be the matrix of D with respect to the b

Let  $A_D$  be the matrix of D with respect to the bases  $1, x, x^2$  and 1, x. Columns of  $A_D$  are coordinates of polynomials D1, Dx,  $Dx^2$  w.r.t. the basis 1, x.

$$D1 = 0$$
,  $Dx = 1$ ,  $Dx^2 = 2x \implies A_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ 

•  $L: \mathcal{P}_3 \to \mathcal{P}_3$ , (Lp)(x) = p(x+1). Let  $A_L$  be the matrix of L w.r.t. the basis  $1, x, x^2$ .

Let  $A_L$  be the matrix of L w.r.t. the basis  $1, x, x^2$ . L1 = 1, Lx = 1 + x,  $Lx^2 = (x + 1)^2 = 1 + 2x + x^2$ .

$$\implies A_L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$