MATH 311

Topics in Applied Mathematics I

Lecture 23:
Eigenvalues and eigenvectors of a linear operator.
Basis of eigenvectors.

Eigenvalues and eigenvectors of a matrix

Definition. Let A be an $n \times n$ matrix. A number $\lambda \in \mathbb{R}$ is called an **eigenvalue** of the matrix A if $\boxed{A\mathbf{v} = \lambda\mathbf{v}}$ for a nonzero column vector $\mathbf{v} \in \mathbb{R}^n$. The vector $\mathbf{v}$ is called an **eigenvector** of A belonging to (or associated with) the eigenvalue λ.

If λ is an eigenvalue of A then the nullspace $N(A - \lambda I)$, which is nontrivial, is called the **eigenspace** of A corresponding to λ. The eigenspace consists of all eigenvectors belonging to the eigenvalue λ plus the zero vector.

Characteristic equation

Definition. Given a square matrix A, the equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

Eigenvalues λ of A are roots of the characteristic equation.

If A is an $n \times n$ matrix then $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n. It is called the **characteristic polynomial** of A.

Theorem Any $n \times n$ matrix has at most n eigenvalues.

Eigenvalues and eigenvectors of an operator

Definition. Let V be a vector space and $L : V \to V$ be a linear operator. A number λ is called an **eigenvalue** of the operator L if $\boxed{L(\mathbf{v}) = \lambda\mathbf{v}}$ for a nonzero vector $\mathbf{v} \in V$. The vector $\mathbf{v}$ is called an **eigenvector** of L associated with the eigenvalue λ. (If V is a functional vector space then eigenvectors are usually called **eigenfunctions**.)

If $V = \mathbb{R}^n$ then the linear operator L is given by $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix (and $\mathbf{x}$ is regarded a column vector). In this case, eigenvalues and eigenvectors of the operator L are precisely eigenvalues and eigenvectors of the matrix A.

Eigenspaces

Let $L : V \to V$ be a linear operator.

For any $\lambda \in \mathbb{R}$, let V_λ denotes the set of all solutions of the equation $L(\mathbf{x}) = \lambda\mathbf{x}$.

Then V_λ is a *subspace* of V since V_λ is the *kernel* of a linear operator given by $\mathbf{x} \mapsto L(\mathbf{x}) - \lambda\mathbf{x}$.

V_λ minus the zero vector is the set of all eigenvectors of L associated with the eigenvalue λ. In particular, $\lambda \in \mathbb{R}$ is an eigenvalue of L if and only if $V_\lambda \neq \{\mathbf{0}\}$.

If $V_\lambda \neq \{\mathbf{0}\}$ then it is called the **eigenspace** of L corresponding to the eigenvalue λ.

Example. $V = C^\infty(\mathbb{R})$, $D : V \to V$, $Df = f'$.

A function $f \in C^\infty(\mathbb{R})$ is an eigenfunction of the operator D belonging to an eigenvalue λ if $f'(x) = \lambda f(x)$ for all $x \in \mathbb{R}$.

It follows that $f(x) = ce^{\lambda x}$, where c is a nonzero constant.

Thus each $\lambda \in \mathbb{R}$ is an eigenvalue of D.
The corresponding eigenspace is spanned by $e^{\lambda x}$.

Example. $V = C^\infty(\mathbb{R})$, $L : V \to V$, $Lf = f''$.

$Lf = \lambda f \iff f''(x) - \lambda f(x) = 0$ for all $x \in \mathbb{R}$.

It follows that each $\lambda \in \mathbb{R}$ is an eigenvalue of L and the corresponding eigenspace V_λ is two-dimensional. Note that $L = D^2$, hence $Df = \mu f \implies Lf = \mu^2 f$.

If $\lambda > 0$ then $V_\lambda = \mathrm{Span}(e^{\mu x}, e^{-\mu x})$, where $\mu = \sqrt{\lambda}$.

If $\lambda < 0$ then $V_\lambda = \mathrm{Span}\big(\sin(\mu x), \cos(\mu x)\big)$, where $\mu = \sqrt{-\lambda}$.

If $\lambda = 0$ then $V_\lambda = \mathrm{Span}(1, x)$.

Suppose $L : V \to V$ is a linear operator on a **finite-dimensional** vector space V.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be a basis for V and $g : V \to \mathbb{R}^n$ be the corresponding coordinate mapping. Let A be the matrix of L with respect to this basis. Then

$$L(\mathbf{v}) = \lambda\mathbf{v} \iff A\,g(\mathbf{v}) = \lambda\,g(\mathbf{v}).$$

Hence the eigenvalues of L coincide with those of the matrix A. Moreover, the associated eigenvectors of A are coordinates of the eigenvectors of L.

Definition. The characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of the matrix A is called the **characteristic polynomial** of the operator L.

Then eigenvalues of L are roots of its characteristic polynomial.

Theorem. The characteristic polynomial of the operator L is well defined. That is, it does not depend on the choice of a basis.

Proof: Let B be the matrix of L with respect to a different basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. Then $A = UBU^{-1}$, where U is the transition matrix from the basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \dots, \mathbf{u}_n$. We have to show that $\det(A - \lambda I) = \det(B - \lambda I)$ for all $\lambda \in \mathbb{R}$. We obtain

$$\det(A - \lambda I) = \det(UBU^{-1} - \lambda I)$$
$$= \det\big(UBU^{-1} - U(\lambda I)U^{-1}\big) = \det\big(U(B - \lambda I)U^{-1}\big)$$
$$= \det(U)\,\det(B - \lambda I)\,\det(U^{-1}) = \det(B - \lambda I).$$

Basis of eigenvectors

Let V be a finite-dimensional vector space and $L : V \to V$ be a linear operator. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and A be the matrix of the operator L with respect to this basis.

Theorem The matrix A is diagonal if and only if vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are eigenvectors of L.

If this is the case, then the diagonal entries of the matrix A are the corresponding eigenvalues of L.

$$L(\mathbf{v}_i) = \lambda_i \mathbf{v}_i \iff A = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix}$$

How to find a basis of eigenvectors

Theorem If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of a linear operator L associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Corollary 1 Suppose $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all eigenvalues of a linear operator $L : V \to V$. For any $1 \leq i \leq k$, let S_i be a basis for the eigenspace associated to the eigenvalue λ_i. Then these bases are disjoint and the union $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent set.

Moreover, if the vector space V admits a basis consisting of eigenvectors of L, then S is such a basis.

Corollary 2 Let A be an $n \times n$ matrix such that the characteristic equation $\det(A - \lambda I) = 0$ has n distinct roots. Then **(i)** there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A; **(ii)** all eigenspaces of A are one-dimensional.

Diagonalization

Theorem 1 Let L be a linear operator on a finite-dimensional vector space V. Then the following conditions are equivalent:

- the matrix of L with respect to some basis is diagonal;
- there exists a basis for V formed by eigenvectors of L.

The operator L is **diagonalizable** if it satisfies these conditions.

Theorem 2 Let A be an $n \times n$ matrix. Then the following conditions are equivalent:

- A is the matrix of a diagonalizable operator;
- A is similar to a diagonal matrix, i.e., it is represented as $A = UBU^{-1}$, where the matrix B is diagonal;
- there exists a basis for $\mathbb{R}^n$ formed by eigenvectors of A.

The matrix A is **diagonalizable** if it satisfies these conditions.

Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 1 and 3.
- The eigenspace of A associated with the eigenvalue 1 is the line spanned by $\mathbf{v}_1 = (-1, 1)$.
- The eigenspace of A associated with the eigenvalue 3 is the line spanned by $\mathbf{v}_2 = (1, 1)$.
- Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$.

Thus the matrix A is diagonalizable. Namely, $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Notice that U is the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2$ to the standard basis.

Example. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 0 and 2.
- The eigenspace for 0 is one-dimensional; it has a basis $S_1 = \{\mathbf{v}_1\}$, where $\mathbf{v}_1 = (-1, 1, 0)$.
- The eigenspace for 2 is two-dimensional; it has a basis $S_2 = \{\mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (-1, 0, 1)$.
- The union $S_1 \cup S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, hence it is a basis for $\mathbb{R}^3$.

Thus the matrix A is diagonalizable. Namely, $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

There are **two obstructions** to existence of a basis consisting of eigenvectors. They are illustrated by the following examples.

Example 1. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\det(A - \lambda I) = (\lambda - 1)^2$. Hence $\lambda = 1$ is the only eigenvalue. The associated eigenspace is the line $t(1, 0)$.

Example 2. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = \lambda^2 + 1$.

$\Longrightarrow$ no real eigenvalues or eigenvectors

(However there are *complex* eigenvalues/eigenvectors.)