MATH 311

Topics in Applied Mathematics I

Lecture 30b:
Review of differential calculus.

Limit of a sequence

Definition. Sequence $x_1, x_2, x_3, \dots$ of real numbers is said to **converge** to a real number a if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq N$. The number a is called the **limit** of $\{x_n\}$.

Notation: $\lim_{n\to\infty} x_n = a$, or $x_n \to a$ as $n \to \infty$.

Note that $d(x, y) = |x - y|$ is the distance between points x and y on the real line.

The condition $|x_n - a| < \varepsilon$ is equivalent to $x_n \in (a - \varepsilon, a + \varepsilon)$. The interval $(a - \varepsilon, a + \varepsilon)$ is called the ε-**neighborhood** of the point a. The convergence $x_n \to a$ means that any ε-neighborhood of a contains all but finitely many elements of the sequence $\{x_n\}$.

Limit of a function

Suppose $f : E \to \mathbb{R}$ is a function defined on a set $E \subset \mathbb{R}$.

Definition. We say that the function f **converges to a limit** $L \in \mathbb{R}$ at a point a if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |x - a| < \delta \text{ implies } |f(x) - L| < \varepsilon.$$

Notation: $L = \lim_{x \to a} f(x)$ or $f(x) \to L$ as $x \to a$.

Theorem Let I be an open interval containing a point $a \in \mathbb{R}$ and f be a function defined on $I \setminus \{a\}$. Then $f(x) \to L$ as $x \to a$ if and only if for any sequence $\{x_n\}$ of elements of $I \setminus \{a\}$,

$$\lim_{n \to \infty} x_n = a \quad \text{implies} \quad \lim_{n \to \infty} f(x_n) = L.$$

Continuity

Definition. Given a set $E \subset \mathbb{R}$, a function $f : E \to \mathbb{R}$, and a point $c \in E$, the function f is **continuous at** c if

$$f(c) = \lim_{x \to c} f(x).$$

That is, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - c| < \delta$ and $x \in E$ imply $|f(x) - f(c)| < \varepsilon$.

Theorem A function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$ if and only if for any sequence $\{x_n\}$ of elements of E, $x_n \to c$ as $n \to \infty$ implies $f(x_n) \to f(c)$ as $n \to \infty$.

We say that the function f is **continuous on** a set $E_0 \subset E$ if f is continuous at every point $c \in E_0$. The function f is **continuous** if it is continuous on the entire domain E.

Topology of the real line

Definition. A sequence $\{x_n\}$ of real numbers is called a **Cauchy sequence** if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ whenever $n, m \geq N$.

Theorem (Cauchy) Any Cauchy sequence is convergent.

This property of $\mathbb{R}$ is called **completeness**.

Theorem (Bolzano-Weierstrass) Every bounded sequence of real numbers has a convergent subsequence.

This property of $\mathbb{R}$ is called **local compactness**.

A set $S \subset \mathbb{R}$ is called **compact** if any sequence of its elements has a subsequence converging to a limit in S. For example, any closed bounded interval $[a, b]$ is compact.

Extreme Value Theorem If $S \subset \mathbb{R}$ is compact, then any continuous function $f : S \to \mathbb{R}$ attains its extreme values on S.

The derivative

Definition. A real function f is said to be **differentiable** at a point $a \in \mathbb{R}$ if it is defined on an open interval containing a and the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The limit is denoted $f'(a)$ and called the **derivative** of f at a. An equivalent condition is

$$f(a+h) = f(a) + f'(a)h + r(h), \text{ where } \lim_{h \to 0} r(h)/h = 0.$$

If a function f is differentiable at a point a, then it is continuous at a.

Suppose that a function f is defined and differentiable on an interval I. Then the derivative of f can be regarded as a function on I. *Notation:* f', $\dot{f}$, $\dfrac{df}{dx}$, $D_x f$, $f^{(1)}$.

Differentiability theorems

Sum Rule If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the sum $f + g$ is also differentiable at a. Moreover, $(f + g)'(a) = f'(a) + g'(a)$.

Homogeneous Rule If a function f is differentiable at a point $a \in \mathbb{R}$, then for any $r \in \mathbb{R}$ the scalar multiple rf is also differentiable at a. Moreover, $(rf)'(a) = rf'(a)$.

Difference Rule If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the difference $f - g$ is also differentiable at a. Moreover, $(f - g)'(a) = f'(a) - g'(a)$.

Differentiability theorems

Product Rule If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the product fg is also differentiable at a. Moreover, $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

Reciprocal Rule If a function f is differentiable at a point $a \in \mathbb{R}$ and $f(a) \neq 0$, then the function $1/f$ is also differentiable at a. Moreover, $(1/f)'(a) = -f'(a)/f^2(a)$.

Quotient Rule If functions f and g are differentiable at $a \in \mathbb{R}$ and $g(a) \neq 0$, then the quotient f/g is also differentiable at a. Moreover,

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

Differentiability theorems

Chain Rule If a function f is differentiable at a point $a \in \mathbb{R}$ and a function g is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a. Moreover, $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Derivative of the inverse function Suppose f is an invertible continuous function. If f is differentiable at a point a and $f'(a) \neq 0$, then the inverse function is differentiable at the point $b = f(a)$ and, moreover,

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

In the case $f'(a) = 0$, the inverse function f^{-1} is not differentiable at $f(a)$.

Properties of differentiable functions

Fermat's Theorem If a function f is differentiable at a point c of local extremum (maximum or minimum), then $f'(c) = 0$.

Rolle's Theorem If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Mean Value Theorem If a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)\,(b - a)$.