MATH 311

Topics in Applied Mathematics I

Lecture 34: Multiple integrals. Line integrals.

Riemann sums in two dimensions

Consider a closed coordinate rectangle
$R = [a, b] \times [c, d] \subset \mathbb{R}^2$.

Definition. A **Riemann sum** of a function $f : R \to \mathbb{R}$ with respect to a partition $P = \{D_1, D_2, \dots, D_n\}$ of R generated by samples $t_j \in D_j$ is a sum

$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^{n} f(t_j)\,\mathrm{area}(D_j).$$

The norm of the partition P is $\|P\| = \max_{1 \le j \le n} \mathrm{diam}(D_j)$.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j.

If this is the case, then the function f is called **integrable** on R and the limit $I(f)$ is called the **integral** of f over R.

Double integral

Closed coordinate rectangle $R = [a, b] \times [c, d]$
$= \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \ \ c \leq y \leq d\}$.

Notation: $\displaystyle\iint_R f \, dA$ or $\displaystyle\iint_R f(x, y) \, dx \, dy$.

Theorem 1 If f is continuous on the closed rectangle R, then f is integrable.

Theorem 2 A function $f : R \to \mathbb{R}$ is Riemann integrable on the rectangle R if and only if f is bounded on R and continuous almost everywhere on R (that is, the set of discontinuities of f has zero area).

Fubini's Theorem

Fubini's Theorem allows us to reduce a multiple integral to a repeated one-dimensional integral.

Theorem If a function f is integrable on $R = [a, b] \times [c, d]$, then

$$\iint_R f\, dA = \int_a^b \Big(\int_c^d f(x, y)\, dy \Big)\, dx = \int_c^d \Big(\int_a^b f(x, y)\, dx \Big)\, dy.$$

In particular, this implies that we can change the order of integration in a repeated integral.

Corollary If a function g is integrable on $[a, b]$ and a function h is integrable on $[c, d]$, then the function $f(x, y) = g(x)h(y)$ is integrable on $R = [a, b] \times [c, d]$ and

$$\iint_R g(x)h(y)\, dx\, dy = \int_a^b g(x)\, dx \cdot \int_c^d h(y)\, dy.$$

Integrals over general domains

Suppose $f : D \to \mathbb{R}$ is a function defined on a (Jordan) measurable set $D \subset \mathbb{R}^2$. Since D is bounded, it is contained in a rectangle R. To define the integral of f over D, we extend the function f to a function on R:

$$f^{\text{ext}}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{if } (x, y) \notin D. \end{cases}$$

Definition. $\iint_D f \, dA$ is defined to be $\iint_R f^{\text{ext}} \, dA$.

In particular, $\mathrm{area}(D) = \iint_D 1 \, dA$.

Integration as a linear operation

Theorem 1 If functions f, g are integrable on a set $D \subset \mathbb{R}^2$, then the sum $f + g$ is also integrable on D and

$$\iint_D (f+g)\,dA = \iint_D f\,dA + \iint_D g\,dA.$$

Theorem 2 If a function f is integrable on a set $D \subset \mathbb{R}^2$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on D and

$$\iint_D \alpha f\,dA = \alpha \iint_D f\,dA.$$

More properties of integrals

Theorem 3 If functions f, g are integrable on a set $D \subset \mathbb{R}^2$, and $f(x,y) \le g(x,y)$ for all $(x,y) \in D$, then

$$\iint_D f\,dA \le \iint_D g\,dA.$$

Theorem 4 If a function f is integrable on sets $D_1, D_2 \subset \mathbb{R}^2$, then it is integrable on their union $D_1 \cup D_2$. Moreover, if the sets D_1 and D_2 are disjoint up to a set of zero area, then

$$\iint_{D_1 \cup D_2} f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA.$$

Change of variables in a double integral

Theorem Let $D \subset \mathbb{R}^2$ be a measurable domain and f be an integrable function on D. If $\mathbf{T} = (u, v)$ is a smooth coordinate mapping such that $\mathbf{T}^{-1}$ is defined on D, then

$$\iint_D f(u, v)\, du\, dv$$

$$= \iint_{\mathbf{T}^{-1}(D)} f\big(u(x, y), v(x, y)\big) \left|\det \frac{\partial(u, v)}{\partial(x, y)}\right| \, dx\, dy.$$

In particular, the integral in the right-hand side is well defined.

Problem Evaluate a double integral

$$\iint_D f(x,y)\,dx\,dy$$

over a disc D bounded by the circle $(x-x_0)^2+(y-y_0)^2=R^2$.

To evaluate the integral, we move the origin to (x_0,y_0) and then switch to polar coordinates (r,ϕ). That is, we use the substitution $(x,y)=T(r,\phi)=(x_0+r\cos\phi,\ y_0+r\sin\phi)$.

Jacobian matrix: $J=\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi}\end{pmatrix}=\begin{pmatrix}\cos\phi & -r\sin\phi\\ \sin\phi & r\cos\phi\end{pmatrix}$.

Then $\det J=r\cos^2\phi+r\sin^2\phi=r$. Hence

$$\iint_D f(x,y)\,dx\,dy=\iint_{T^{-1}(D)} f(x_0+r\cos\phi,\ y_0+r\sin\phi)\,|\det J|\,dr\,d\phi$$

$$=\int_0^{2\pi}\int_0^R f(x_0+r\cos\phi,\ y_0+r\sin\phi)\,r\,dr\,d\phi.$$

Problem Evaluate a double integral

$$\iint_P f(x,y)\,dx\,dy$$

over a parallelogram P with vertices $(-1,-1)$, $(1,0)$, $(2,2)$, and $(0,1)$.

Adjacent edges of the parallelogram P are represented by vectors $\mathbf{v}_1 = (1,0) - (-1,-1) = (2,1)$ and $\mathbf{v}_2 = (0,1) - (-1,-1) = (1,2)$.

Consider a transformation L of the plane $\mathbb{R}^2$ given by

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2u + v - 1 \\ u + 2v - 1 \end{pmatrix}$$

(columns of the matrix are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$). By construction, L maps the unit square $[0,1]\times[0,1]$ onto the parallelogram P. The Jacobian matrix J of L is the same at any point: $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Changing coordinates in the integral from (x, y) to (u, v) so that

$$(x, y) = L(u, v) = (2u + v - 1,\ u + 2v - 1),$$

we obtain

$$\iint_P f(x, y)\, dx\, dy$$

$$= \iint_{L^{-1}(P)} f(2u + v - 1,\ u + 2v - 1)\, |\det J|\, du\, dv$$

$$= 3 \int_0^1 \int_0^1 f(2u + v - 1,\ u + 2v - 1)\, du\, dv.$$

Triple integral

To integrate in $\mathbb{R}^3$, volumes are used instead of areas in $\mathbb{R}^2$. Instead of coordinate rectangles, basic sets are coordinate boxes (or bricks) $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$. Then we can define an integral of a function f over a measurable set $D \subset \mathbb{R}^3$.

Notation: $\iiint_D f \, dV$ or $\iiint_D f(x, y, z) \, dx \, dy \, dz$.

The properties of triple integrals are completely analogous to those of double integrals. In particular, Fubini's Theorem is formulated as follows.

Theorem If a function f is integrable on a brick $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$, then

$$\iiint_B f \, dV = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \left(\int_{a_3}^{b_3} f(x, y, z) \, dz \right) dy \right) dx.$$

Path

Definition. A **path** in $\mathbb{R}^n$ is a continuous function $\mathbf{x} : [a, b] \to \mathbb{R}^n$.

Paths provide parametrizations for curves.

Length of the path $\mathbf{x}$ is defined as $L = \sup_P \sum_{j=1}^{k} \|\mathbf{x}(t_j) - \mathbf{x}(t_{j-1})\|$ over all partitions $P = \{t_0, t_1, \ldots, t_k\}$ of the interval $[a, b]$.

Theorem The length of a smooth path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is $\displaystyle\int_a^b \|\mathbf{x}'(t)\|\, dt$.

Arclength parameter: $\displaystyle s(t) = \int_a^t \|\mathbf{x}'(\tau)\|\, d\tau$.

Scalar line integral

Scalar line integral is an integral of a scalar function f over a path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ of finite length relative to the arclength. It is defined as a limit of Riemann sums

$$\mathcal{S}(f, P, \tau_j) = \sum_{j=1}^{k} f(\mathbf{x}(\tau_j)) \left(s(t_j) - s(t_{j-1})\right),$$

where $P = \{t_0, t_1, \ldots, t_k\}$ is a partition of $[a, b]$, $\tau_j \in [t_j, t_{j-1}]$ for $1 \leq j \leq k$, and s is the arclength parameter of the path $\mathbf{x}$.

Theorem Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a smooth path and f be a function defined on the image of this path. Then

$$\int_{\mathbf{x}} f \, ds = \int_a^b f(\mathbf{x}(t)) \, \|\mathbf{x}'(t)\| \, dt.$$

ds is referred to as the arclength element.

Vector line integral

Vector line integral is an integral of a vector field over a smooth path. It is a scalar.

Definition. Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a smooth path and $\mathbf{F}$ be a vector field defined on the image of this path. Then $\displaystyle\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt$.

Alternatively, the integral of $\mathbf{F}$ over $\mathbf{x}$ can be represented as the integral of a **differential form**

$$\int_{\mathbf{x}} F_1\, dx_1 + F_2\, dx_2 + \cdots + F_n\, dx_n,$$

where $\mathbf{F} = (F_1, F_2, \dots, F_n)$ and $dx_i = x_i'(t)\, dt$.

Applications of line integrals

- Mass of a wire

If f is the density on a wire C, then $\int_C f\, ds$ is the mass of C.

- Work of a force

If $\mathbf{F}$ is a force field, then $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ is the work done by $\mathbf{F}$ on a particle that moves along the path $\mathbf{x}$.

- Circulation of fluid

If $\mathbf{F}$ is the velocity field of a planar fluid, then the circulation of the fluid across a closed curve C is $\oint_C \mathbf{F} \cdot d\mathbf{s}$.

- Flux of fluid

If $\mathbf{F}$ is the velocity field of a planar fluid, then the flux of the fluid across a closed curve C is $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$, where $\mathbf{n}$ is the outward unit normal vector to C.