MATH 311

Topics in Applied Mathematics I

Lecture 36: Conservative vector fields. Area of a surface.

Conservative vector fields

Let R be an open region in $\mathbb{R}^n$ such that any two points in R can be connected by a continuous path. Such regions are called **(arcwise) connected**.

Definition. A continuous vector field $\mathbf{F} : R \to \mathbb{R}^n$ is called **conservative** if $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$ for any two simple, piecewise smooth, oriented curves $C_1, C_2 \subset R$ with the same initial and terminal points.

An equivalent condition is that $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$ for any piecewise smooth closed curve $C \subset R$.

Conservative vector fields

Theorem The vector field $\mathbf{F}$ is conservative if and only if it is a gradient field, that is, $\mathbf{F} = \nabla f$ for some function $f : R \to \mathbb{R}$. If this is the case, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A)$$

for any piecewise smooth, oriented curve $C \subset R$ that connects the point A to the point B.

Remark. In the case $\mathbf{F}$ is a force field, conservativity means that energy is conserved. Moreover, in this case the function f is the potential energy.

Test of conservativity

Theorem If a smooth field $\mathbf{F} = (F_1, F_2, \dots, F_n)$ is conservative in a region $R \subset \mathbb{R}^n$, then the Jacobian matrix $\dfrac{\partial(F_1, F_2, \dots, F_n)}{\partial(x_1, x_2, \dots, x_n)}$ is symmetric everywhere in R, that is,

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \text{ for } i \neq j.$$

Indeed, if the field $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla f$ for some smooth function $f : R \to \mathbb{R}$. It follows that the Jacobian matrix of $\mathbf{F}$ is the **Hessian matrix** of f, that is, the matrix of second-order partial derivatives: $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial^2 f}{\partial x_j \, \partial x_i}$.

Remark. The converse of the theorem holds provided that the region R is **simply-connected**, which means that any closed path in R can be continuously shrunk within R to a point.

Finding scalar potential

Example. $\mathbf{F}(x, y) = (2xy^3 + 3y\cos 3x,\ 3x^2y^2 + \sin 3x)$.

The vector field $\mathbf{F}$ is conservative if $\partial F_1/\partial y = \partial F_2/\partial x$.

$$\frac{\partial F_1}{\partial y} = 6xy^2 + 3\cos 3x, \quad \frac{\partial F_2}{\partial x} = 6xy^2 + 3\cos 3x.$$

Thus $\mathbf{F} = \nabla f$ for some function f (**scalar potential** of $\mathbf{F}$),

that is, $\dfrac{\partial f}{\partial x} = 2xy^3 + 3y\cos 3x, \quad \dfrac{\partial f}{\partial y} = 3x^2y^2 + \sin 3x.$

Integrating the second equality by y, we get

$$f(x, y) = \int (3x^2y^2 + \sin 3x)\, dy = x^2y^3 + y\sin 3x + g(x).$$

Substituting this into the first equality, we obtain that $2xy^3 + 3y\cos 3x + g'(x) = 2xy^3 + 3y\cos 3x$. Hence $g'(x) = 0$ so that $g(x) = c$, a constant. Then $f(x, y) = x^2y^3 + y\sin 3x + c$.

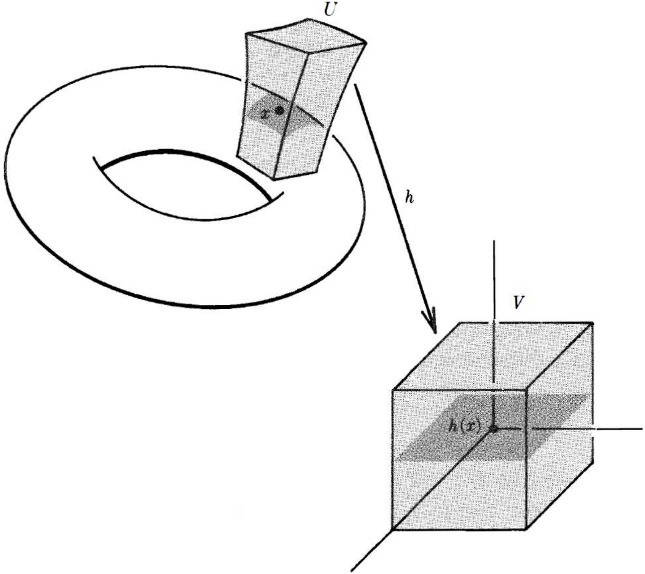
Surface

Suppose D_1 and D_2 are domains in $\mathbb{R}^3$ and $\mathbf{T} : D_1 \to D_2$ is an invertible map such that both $\mathbf{T}$ and $\mathbf{T}^{-1}$ are smooth. Then we say that $\mathbf{T}$ defines **curvilinear coordinates** in D_1.

Definition. A nonempty set $S \subset \mathbb{R}^3$ is called a **smooth surface** if for every point $\mathbf{p} \in S$ there exist curvilinear coordinates $\mathbf{T} : D_1 \to D_2$ in a neighborhood of $\mathbf{p}$ such that $\mathbf{T}(\mathbf{p}) = \mathbf{0}$ and either $\mathbf{T}(S \cap D_1) = \{(x, y, z) \in D_2 \mid z = 0\}$ or $\mathbf{T}(S \cap D_1) = \{(x, y, z) \in D_2 \mid z = 0,\ y \geq 0\}$. In the first case, $\mathbf{p}$ is called an **interior point** of the surface S, in the second case, $\mathbf{p}$ is called a **boundary point** of S.

The set of all boundary points of the surface S is called the **boundary** of S and denoted ∂S.

A smooth surface S is called **complete** if for any convergent sequence of points from S, the limit belongs to S as well. A complete surface with no boundary points is called **closed**.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Parametrized surfaces

Definition. Let $D \subset \mathbb{R}^2$ be a connected, bounded region. A continuous one-to-one map $\mathbf{X} : D \to \mathbb{R}^3$ is called a **parametrized surface**. The image $\mathbf{X}(D)$ is called the **underlying surface**.

The parametrized surface is **smooth** if $\mathbf{X}$ is smooth and, moreover, the vectors $\frac{\partial \mathbf{X}}{\partial s}(s_0, t_0)$ and $\frac{\partial \mathbf{X}}{\partial t}(s_0, t_0)$ are linearly independent for all $(s_0, t_0) \in D$. If this is the case, then the plane in $\mathbb{R}^3$ through the point $\mathbf{X}(s_0, t_0)$ parallel to vectors $\frac{\partial \mathbf{X}}{\partial s}(s_0, t_0)$ and $\frac{\partial \mathbf{X}}{\partial t}(s_0, t_0)$ is called the **tangent plane** to $\mathbf{X}(D)$ at $\mathbf{X}(s_0, t_0)$.

Example. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and consider a **level set** $P = \{(x, y, z) : f(x, y, z) = c\}$, $c \in \mathbb{R}$. If $\nabla f \neq \mathbf{0}$ at some point $p \in P$, then near that point P is the underlying surface of a parametrized surface. Moreover, the gradient $(\nabla f)(p)$ is orthogonal to the tangent plane at p.

Plane in space

Consider a map $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{X}\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}.$$

Partial derivatives $\frac{\partial \mathbf{X}}{\partial s}$ and $\frac{\partial \mathbf{X}}{\partial t}$ are constant, namely, they are columns of the matrix $A = (a_{ij})$. Assume that the columns are linearly independent. Then $\mathbf{X}$ is a parametrized surface. The underlying surface is a plane Π. The tangent plane at every point is Π itself.

For a measurable set $D \subset \mathbb{R}^2$, the image $\mathbf{X}(D)$ is measurable in the plane Π. Moreover, $\mathrm{area}\big(\mathbf{X}(D)\big) = \alpha\, \mathrm{area}(D)$ for some fixed scalar α. To determine α, consider the unit square $Q = [0,1] \times [0,1]$. The image $\mathbf{X}(Q)$ is a parallelogram with adjacent sides represented by vectors $\frac{\partial \mathbf{X}}{\partial s}$ and $\frac{\partial \mathbf{X}}{\partial t}$. We obtain that $\alpha = \mathrm{area}\big(\mathbf{X}(Q)\big) = \|\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t}\|$.

Area of a surface

Let P be a smooth surface parametrized by $\mathbf{X} : D \to \mathbb{R}^3$. Then the area of P is

$$\text{area}(P) = \iint_D \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| ds\, dt.$$

Suppose P is the graph of a smooth function $g : D \to \mathbb{R}$, i.e., P is given by $z = g(x, y)$. We have a natural parametrization $\mathbf{X} : D \to \mathbb{R}^3$, $\mathbf{X}(x, y) = (x, y, g(x, y))$. Then $\frac{\partial \mathbf{X}}{\partial x} = (1, 0, g'_x)$ and $\frac{\partial \mathbf{X}}{\partial y} = (0, 1, g'_y)$. Consequently,

$$\frac{\partial \mathbf{X}}{\partial x} \times \frac{\partial \mathbf{X}}{\partial y} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & g'_x \\ 0 & 1 & g'_y \end{vmatrix} = (-g'_x, -g'_y, 1).$$

It follows that

$$\text{area}(P) = \iint_D \sqrt{1 + |g'_x|^2 + |g'_y|^2}\, dx\, dy.$$