MATH 311

Topics in Applied Mathematics I

Lecture 38:
Review for Test 3.

Topics for Test 3

Vector analysis (Leon/Colley 8.1–8.4, 9.1–9.5, 10.1–10.3, 11.1–11.3)

- Gradient, divergence, and curl

- Fubini's Theorem
- Change of coordinates in a multiple integral

- Length of a curve
- Line integrals
- Green's Theorem
- Conservative vector fields

- Area of a surface
- Surface integrals
- Gauss' Theorem
- Stokes' Theorem

Sample problems for Test 3

Problem 1 Find $\mathrm{curl}(\mathrm{curl}(\mathbf{F}))$, where $\mathbf{F}(x, y, z) = (x^2 + y^2)\mathbf{e}_1 + ze^{x+y}\mathbf{e}_2 + (x + \sin y)\mathbf{e}_3$.

Problem 2 Evaluate a double integral

$$\iint_P \big(2x + 3y - \cos(\pi x + 2\pi y)\big)\,dx\,dy$$

over a parallelogram P with vertices $(-1, -1)$, $(1, 0)$, $(2, 2)$, and $(0, 1)$.

Sample problems for Test 3

Problem 3 Find the volume of a tetrahedron (i.e., triangular pyramid) with vertices at points $(0, 2, 1)$, $(1, 0, 0)$, $(2, 1, 2)$, and $(3, 1, 1)$.

Problem 4 Consider a vector field
$\mathbf{F}(x, y, z) = (yz + 2\cos 2x,\ xz - e^z,\ xy - ye^z)$.

(i) Verify that the field $\mathbf{F}$ is conservative.
(ii) Find a function f such that $\mathbf{F} = \nabla f$.

Sample problems for Test 3

Problem 5 Let C be a solid cylinder bounded by planes $z = 0$, $z = 2$ and a cylindrical surface $x^2 + y^2 = 1$. Orient the boundary ∂C with outward normals and evaluate a surface integral

$$\oiint_{\partial C} (x^2\mathbf{e}_1 + y^2\mathbf{e}_2 + z^2\mathbf{e}_3) \cdot d\mathbf{S}.$$

Problem 6 Let D be a region in $\mathbb{R}^3$ bounded by a paraboloid $z = x^2 + y^2$ and a plane $z = 9$. Let S denote the part of the paraboloid that bounds D, oriented by outward normals. Evaluate a surface integral

$$\iint_S \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = (e^{x^2+z^2},\ xy + xz + yz,\ e^{xyz})$.

Problem 1 Find $\mathrm{curl}(\mathrm{curl}(\mathbf{F}))$, where $\mathbf{F}(x,y,z) = (x^2+y^2)\mathbf{e}_1 + ze^{x+y}\mathbf{e}_2 + (x+\sin y)\mathbf{e}_3$.

For any vector field $\mathbf{F} = (F_1, F_2, F_3)$ we have, informally,

$$\mathrm{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

or, formally,

$$\mathrm{curl}\,\mathbf{F} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big).$$

Problem 1 Find $\operatorname{curl}(\operatorname{curl}(\mathbf{F}))$, where $\mathbf{F}(x, y, z) = (x^2 + y^2)\mathbf{e}_1 + ze^{x+y}\mathbf{e}_2 + (x + \sin y)\mathbf{e}_3$.

Let $\mathbf{G} = \operatorname{curl}\mathbf{F}$, $\mathbf{G} = (G_1, G_2, G_3)$. We obtain

$$G_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial}{\partial y}(x + \sin y) - \frac{\partial}{\partial z}(ze^{x+y}) = \cos y - e^{x+y},$$

$$G_2 = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = \frac{\partial}{\partial z}(x^2 + y^2) - \frac{\partial}{\partial x}(x + \sin y) = -1,$$

$$G_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(ze^{x+y}) - \frac{\partial}{\partial y}(x^2 + y^2) = ze^{x+y} - 2y.$$

Hence $\mathbf{G} = \mathrm{curl}\,\mathbf{F} = (\cos y - e^{x+y},\ -1,\ ze^{x+y} - 2y)$.

Now let $\mathbf{H} = \mathrm{curl}\,\mathbf{G}$, $\mathbf{H} = (H_1, H_2, H_3)$. We obtain

$$H_1 = \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = \frac{\partial}{\partial y}(ze^{x+y} - 2y) - \frac{\partial}{\partial z}(-1) = ze^{x+y} - 2,$$

$$H_2 = \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} = \frac{\partial}{\partial z}(\cos y - e^{x+y}) - \frac{\partial}{\partial x}(ze^{x+y} - 2y) = -ze^{x+y},$$

$$H_3 = \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \frac{\partial}{\partial x}(-1) - \frac{\partial}{\partial y}(\cos y - e^{x+y}) = \sin y + e^{x+y}.$$

Thus $\mathrm{curl}(\mathrm{curl}(\mathbf{F})) = (ze^{x+y} - 2,\ -ze^{x+y},\ \sin y + e^{x+y})$.

Problem 2 Evaluate a double integral

$$\iint_P \big(2x + 3y - \cos(\pi x + 2\pi y)\big)\, dx\, dy$$

over a parallelogram P with vertices $(-1,-1)$, $(1,0)$, $(2,2)$, and $(0,1)$.

Let us change coordinates in this integral so that the domain of integration becomes the unit square $Q = [0,1] \times [0,1]$. We are going to use a substitution of the form

$$(x,y) = L(u,v) = (a_{11}u + a_{12}v + b_1,\ a_{21}u + a_{22}v + b_2),$$

where a_{ij}, b_i are constants. The constants are determined from the conditions $L(0,0) = (-1,-1)$, $L(1,0) = (1,0)$, and $L(0,1) = (0,1)$. That is, $(b_1, b_2) = (-1,-1)$, $(a_{11}+b_1, a_{21}+b_2) = (1,0)$, and $(a_{12}+b_1, a_{22}+b_2) = (0,1)$. We obtain that

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u + v - 1 \\ u + 2v - 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

The Jacobian matrix J of L is constant: $J = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$

Changing coordinates in the integral from (x, y) to (u, v) so that $(x, y) = L(u, v) = (2u + v - 1,\ u + 2v - 1)$, we obtain

$$\iint_P \big(2x + 3y - \cos(\pi x + 2\pi y)\big)\, dx\, dy$$

$$= \iint_{L^{-1}(P)} \big(7u + 8v - 5 - \cos(4\pi u + 5\pi v - 3\pi)\big)\, |\det J|\, du\, dv$$

$$= \int_0^1 \int_0^1 3\big(7u + 8v - 5 + \cos(4\pi u + 5\pi v)\big)\, du\, dv$$

$$= \frac{21}{2} + 12 - 15 + \int_0^1 \int_0^1 3\cos(4\pi u + 5\pi v)\, du\, dv.$$

Further, $\displaystyle\int_0^1 3\cos(4\pi u + 5\pi v)\, du = \frac{3}{4\pi} \sin(4\pi u + 5\pi v)\,\Big|_{u=0}^{1}$

$$= \frac{3}{4\pi}\big(\sin(4\pi + 5\pi v) - \sin(5\pi v)\big) = 0 \quad \text{for all } v.$$

It follows that $\displaystyle\iint_P \big(2x + 3y - \cos(\pi x + 2\pi y)\big)\, dx\, dy = \frac{15}{2}.$

Problem 3 Find the volume of a tetrahedron (i.e., triangular pyramid) with vertices at points $(0,2,1)$, $(1,0,0)$, $(2,1,2)$, and $(3,1,1)$.

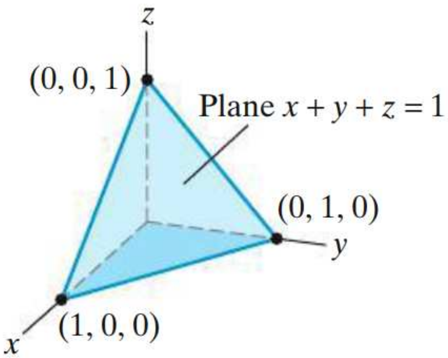
Let P denote the pyramid. Let $A_0 = (1,0,0)$, $A_1 = (0,2,1)$, $A_2 = (2,1,2)$ and $A_3 = (3,1,1)$. Three edges adjacent to A_0 are represented by vectors

$$\mathbf{v}_1 = \overrightarrow{A_0A_1} = (0,2,1) - (1,0,0) = (-1,2,1),$$
$$\mathbf{v}_2 = \overrightarrow{A_0A_2} = (2,1,2) - (1,0,0) = (1,1,2),$$
$$\mathbf{v}_3 = \overrightarrow{A_0A_3} = (3,1,1) - (1,0,0) = (2,1,1).$$

Consider a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The matrix is $M = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.


z
(0, 0, 1)
Plane x + y + z = 1
(0, 1, 0)
y
x
(1, 0, 0)

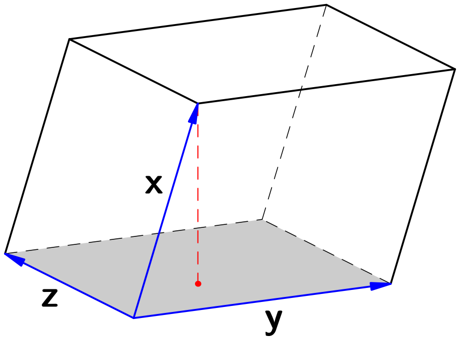
By construction, $T(0,0,0) = A_0$, $T(1,0,0) = A_1$, $T(0,1,0) = A_2$ and $T(0,0,1) = A_3$. It follows that $T^{-1}(P)$ is the triangular pyramid with vertices at points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Consider $(0,0,1)$ to be the apex of the pyramid $T^{-1}(P)$. Then the base is an isosceles right triangle with legs of length 1. Its area equals $\frac{1}{2}$. Besides, the edge $(0,0,0)$–$(0,0,1)$ is the altitude. Therefore the volume of the pyramid $T^{-1}(P)$ equals $\frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$.

We have $\mathrm{volume}(T(D)) = |\det M| \, \mathrm{volume}(D)$ for any domain $D \subset \mathbb{R}^3$. In particular, $\mathrm{volume}(P) = |\det M|/6$.

$$\det M = \begin{vmatrix} -1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{vmatrix} = 6.$$

Thus $\mathrm{volume}(P) = 6 \cdot \frac{1}{6} = 1$.

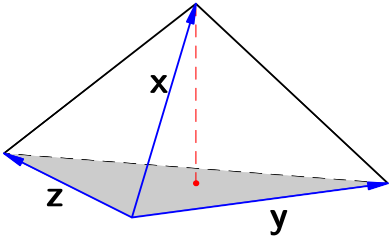


Parallelepiped is a prism.

(Volume) = (area of the base) $\times$ (height)

Area of the base $= \|\mathbf{y} \times \mathbf{z}\|$

Volume $= |\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$



Tetrahedron is a pyramid.

$$(\text{Volume}) = \frac{1}{3}(\text{area of the base}) \times (\text{height})$$

$$\text{Area of the base} = \frac{1}{2}\|\mathbf{y} \times \mathbf{z}\|$$

$$\implies \quad \text{Volume} = \frac{1}{6}|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$$

Problem 4 Consider a vector field
$\mathbf{F}(x, y, z) = (yz + 2\cos 2x,\ xz - e^z,\ xy - ye^z)$.
(i) Verify that the field $\mathbf{F}$ is conservative.

Since $\mathbf{F}$ is a smooth vector field on the entire space, it is conservative if and only if its Jacobian matrix is symmetric everywhere in $\mathbb{R}^3$. For vector fields on $\mathbb{R}^3$, this is equivalent to $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$. We have to verify three identities.

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}: \quad \frac{\partial}{\partial y}(yz+2\cos 2x) = \frac{\partial}{\partial x}(xz-e^z) \iff z = z,$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}: \quad \frac{\partial}{\partial z}(yz+2\cos 2x) = \frac{\partial}{\partial x}(xy-ye^z) \iff y = y,$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}: \quad \frac{\partial}{\partial z}(xz - e^z) = \frac{\partial}{\partial y}(xy - ye^z)$$

$$\iff x - e^z = x - e^z.$$

Problem 4 Consider a vector field
$\mathbf{F}(x, y, z) = (yz + 2\cos 2x,\ xz - e^z,\ xy - ye^z)$.
(ii) Find a function f such that $\mathbf{F} = \nabla f$.

We are looking for a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that
$$\frac{\partial f}{\partial x} = yz + 2\cos 2x, \quad \frac{\partial f}{\partial y} = xz - e^z, \quad \frac{\partial f}{\partial z} = xy - ye^z.$$
Integrating the first equality by x, we get
$$f(x, y, z) = \int (yz + 2\cos 2x)\, dx = xyz + \sin 2x + g(y, z).$$
Substituting this into the second equality, we obtain
$xz + g'_y = xz - e^z$ so that $g'_y = -e^z$. Integrating by y, we get
$$g(y, z) = \int -e^z\, dy = -ye^z + h(z).$$
Then $f(x, y, z) = xyz + \sin 2x - ye^z + h(z)$. Substituting this into the third equality, we obtain $xy - ye^z + h'(z) = xy - ye^z$. Hence $h'(z) = 0$ so that $h(z) = c$, a constant. Finally,
$f(x, y, z) = xyz + \sin 2x - ye^z + c$.

Problem 4 Consider a vector field
$\mathbf{F}(x, y, z) = (yz + 2\cos 2x,\ xz - e^z,\ xy - ye^z)$.
(ii) Find a function f such that $\mathbf{F} = \nabla f$.

Alternative solution: If $\mathbf{F} = \nabla f$, then
$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = f(A_1) - f(A_0)$$
for any points $A_0, A_1 \in \mathbb{R}^3$ and any path $\mathbf{x}$ joining A_0 to A_1. We can use this relation to recover the function f.

For any given point $A = (x, y, z)$ we consider a linear path $\mathbf{x}_A$ from the origin to A, $\mathbf{x}_A : [0, 1] \to \mathbb{R}^3$, $\mathbf{x}_A(t) = (tx, ty, tz)$. Then
$$f(A) - f(\mathbf{0}) = \int_{\mathbf{x}_A} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{x}_A(t)) \cdot \mathbf{x}_A'(t)\, dt.$$

$$f(A) - f(\mathbf{0}) = \int_{\mathbf{x}_A} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{x}_A(t)) \cdot \mathbf{x}_A'(t)\, dt$$

$$= \int_0^1 (t^2yz + 2\cos 2tx,\ t^2xz - e^{tz},\ t^2xy - tye^{tz}) \cdot (x, y, z)\, dt$$

$$= \int_0^1 \big((t^2yz + 2\cos 2tx)x + (t^2xz - e^{tz})y + (t^2xy - tye^{tz})z\big)\, dt$$

$$= \int_0^1 (3t^2xyz + 2x\cos 2tx - ye^{tz} - tyze^{tz})\, dt$$

$$= t^3xyz\,\Big|_{t=0}^1 + \sin 2tx\,\Big|_{t=0}^1 - yte^{tz}\,\Big|_{t=0}^1 = xyz + \sin 2x - ye^z.$$

Thus $f(x, y, z) = xyz + \sin 2x - ye^z + c$, where $c = f(\mathbf{0})$ is a constant.

Problem 5 Let C be a solid cylinder bounded by planes $z = 0$, $z = 2$ and a cylindrical surface $x^2 + y^2 = 1$. Orient the boundary ∂C with outward normals and evaluate a surface integral

$$\oiint_{\partial C} (x^2 \mathbf{e}_1 + y^2 \mathbf{e}_2 + z^2 \mathbf{e}_3) \cdot d\mathbf{S}.$$

By Gauss' Theorem,

$$\oiint_{\partial C} (x^2 \mathbf{e}_1 + y^2 \mathbf{e}_2 + z^2 \mathbf{e}_3) \cdot d\mathbf{S} = \iiint_C \nabla \cdot (x^2 \mathbf{e}_1 + y^2 \mathbf{e}_2 + z^2 \mathbf{e}_3)\, dV$$

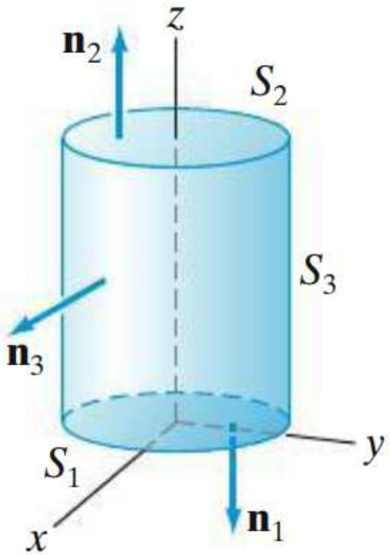
$$= \iiint_C \left(\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) \right) dx\, dy\, dz$$

$$= \iiint_C 2(x + y + z)\, dx\, dy\, dz.$$

To evaluate the integral, we switch to cylindrical coordinates (r, ϕ, z) using the substitution $x = r\cos\phi$, $y = r\sin\phi$, $z = z$.

Jacobian matrix $J = \dfrac{\partial(x, y, z)}{\partial(r, \phi, z)} = \begin{pmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$\begin{aligned}
&\iiint_C 2(x + y + z)\, dx\, dy\, dz \\
&\quad = \int_0^2 \int_0^{2\pi} \int_0^1 2(r\cos\phi + r\sin\phi + z)\big|\det J\big|\, dr\, d\phi\, dz \\
&\quad = \int_0^2 \int_0^{2\pi} \int_0^1 2(r\cos\phi + r\sin\phi + z)\, r\, dr\, d\phi\, dz \\
&\quad = \int_0^2 \int_0^{2\pi} \int_0^1 \Big(2r^2(\cos\phi + \sin\phi) + 2rz\Big)\, dr\, d\phi\, dz \\
&= \int_0^2 \int_0^{2\pi} \int_0^1 2rz\, dr\, d\phi\, dz = 2\int_0^2 z\, dz \cdot \int_0^{2\pi} d\phi \cdot \int_0^1 r\, dr = 4\pi.
\end{aligned}$$


z
n2
S2
S3
n3
y
S1
x
n1

Alternative evaluation of the triple integral:
Consider an invertible linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(x, y, z) = (-x, -y, z)$. The matrix of L (relative to the standard basis) is

$$M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is also the Jacobian matrix of L at every point. Changing coordinates from (x, y, z) to (u, v, w) so that $(x, y, z) = L(u, v, w)$, we obtain

$$\iiint_C 2(x+y)\,dx\,dy\,dz = \iiint_{L^{-1}(C)} 2(-u-v)\,|\det M|\,du\,dv\,dw$$

$$= -\iiint_C 2(u+v)\,du\,dv\,dw.$$

It follows that $\iiint_C 2(x+y)\,dx\,dy\,dz = 0.$

By linearity of the integral,

$$\oiint_{\partial C} (x^2\mathbf{e}_1 + y^2\mathbf{e}_2 + z^2\mathbf{e}_3)\cdot d\mathbf{S} = \iiint_C 2(x+y+z)\,dx\,dy\,dz$$

$$= \iiint_C 2(x+y)\,dx\,dy\,dz + \iiint_C 2z\,dx\,dy\,dz = \iiint_C 2z\,dx\,dy\,dz.$$

The cylinder C can be represented as $C = U \times [0,2]$, where U is the unit disc in the plane,

$$U = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

By Fubini's Theorem,

$$\begin{aligned}\iiint_C 2z\,dx\,dy\,dz &= \iint_U \left(\int_0^2 2z\,dz\right) dx\,dy \\ &= \iint_U 4\,dx\,dy = 4\ \mathrm{area}(U) = 4\pi.\end{aligned}$$

Problem 6 Let D be a region in $\mathbb{R}^3$ bounded by a paraboloid $z = x^2 + y^2$ and a plane $z = 9$. Let S denote the part of the paraboloid that bounds D, oriented by outward normals. Evaluate a surface integral

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = (e^{x^2+z^2},\ xy + xz + yz,\ e^{xyz})$.

We have $\operatorname{curl} \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

$= (xze^{xyz} - x - y,\ 2ze^{x^2+z^2} - yze^{xyz},\ y + z)$.

Direct evaluation of the surface integral seems problematic. By Stokes' Theorem, the surface integral equals the integral of the field $\mathbf{F}$ along the circle ∂S. However evaluation of this line integral seems problematic as well.

By the corollary of Stokes' Theorem,

$$\iint_{\partial D} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = 0.$$

It follows that

$$\iint_{S} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = -\iint_{\partial D\setminus S} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S}.$$

We observe that $\partial D \setminus S$ is a horizontal disc $Q\times\{9\}$, where $Q = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 9\}$. It is oriented by the upward normal vector $\mathbf{n} = (0,0,1)$. Now

$$\iint_{\partial D\setminus S} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_{\partial D\setminus S} \operatorname{curl}(\mathbf{F})\cdot\mathbf{n}\, dS = \iint_{Q} (y+9)\, dx\, dy$$

$$= \iint_Q y\, dx\, dy + \iint_Q 9\, dx\, dy = \iint_Q 9\, dx\, dy = 9\,\mathrm{area}(Q) = 81\pi.$$

Thus $\displaystyle\iint_{S} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = -81\pi.$