MATH 311

Topics in Applied Mathematics I

Lecture 40:
Review for the final exam (continued).

Topics for the final exam: Part I

Elementary linear algebra (L/C 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (L/C 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.

- Linear transformations.
- Matrix transformations.
- Matrix of a linear mapping.
- Change of basis for a linear operator.
- Similarity of matrices.

Topics for the final exam: Part III

Advanced linear algebra (L/C 5.1–5.6, 6.1, 6.3)

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Bases of eigenvectors, diagonalization

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product)
- Inner products and norms
- Orthogonal complement, orthogonal projection
- Least squares problems
- The Gram-Schmidt orthogonalization process

Topics for the final exam: Part IV

Vector analysis (L/C 8.1–8.4, 9.1–9.5, 10.1–10.3, 11.1–11.3)

- Gradient, divergence, and curl
- Fubini's Theorem
- Change of coordinates in a multiple integral
- Geometric meaning of the determinant
- Length of a curve
- Line integrals
- Green's Theorem
- Conservative vector fields
- Area of a surface
- Surface integrals
- Gauss' Theorem
- Stokes' Theorem

Problem. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$, where $\mathbf{v}_0 = (3/5, 0, -4/5)$.

(a) Find the matrix B of the operator L.
(b) Find the range and kernel of L.
(c) Find the eigenvalues of L.
(d) Find the matrix of the operator L^{2020} (L applied 2020 times).

$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \quad \mathbf{v}_0 = (3/5, 0, -4/5).$

Let $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Then

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4/5 \\ y & z \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} 3/5 & -4/5 \\ x & z \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} 3/5 & 0 \\ x & y \end{vmatrix} \mathbf{e}_3$$

$$= \tfrac{4}{5}y\mathbf{e}_1 - \left(\tfrac{4}{5}x + \tfrac{3}{5}z\right)\mathbf{e}_2 + \tfrac{3}{5}y\mathbf{e}_3 = \left(\tfrac{4}{5}y, -\tfrac{4}{5}x - \tfrac{3}{5}z, \tfrac{3}{5}y\right).$$

In particular, $L(\mathbf{e}_1) = \left(0, -\frac{4}{5}, 0\right)$, $L(\mathbf{e}_2) = \left(\frac{4}{5}, 0, \frac{3}{5}\right)$, $L(\mathbf{e}_3) = \left(0, -\frac{3}{5}, 0\right)$.

Therefore $B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}$.

The range of the operator L is spanned by columns of the matrix B. It follows that $\mathrm{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0,1,0)$ and $\mathbf{v}_2 = (4,0,3)$.

The kernel of L is the nullspace of the matrix B, i.e., the solution set for the equation $B\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies x + \tfrac{3}{4}z = y = 0 \implies \mathbf{x} = t(-3/4, 0, 1).$$

Alternatively, the kernel of L is the set of vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$.
It follows that this is the line spanned by $\mathbf{v}_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix B:

$$\det(B - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

The eigenvalues are 0, i, and $-i$.

The matrix of the operator L^{2020} is B^{2020}.

Since the matrix B has eigenvalues 0, i, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $B = UDU^{-1}$, where U is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then $B^{2020} = UD^{2020}U^{-1}$. We have that $D^{2020} =$ $= \mathrm{diag}\big(0, i^{2020}, (-i)^{2020}\big) = \mathrm{diag}(0,1,1) = -D^2$.

Hence

$$B^{2020} = U(-D^2)U^{-1} = -B^2 = \begin{pmatrix} 0.64 & 0 & 0.48 \\ 0 & 1 & 0 \\ 0.48 & 0 & 0.36 \end{pmatrix}.$$

Problem. Find the distance from the point $\mathbf{y} = (0,0,0,1)$ to the subspace $V \subset \mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1,-1,1,-1)$, $\mathbf{x}_2 = (1,1,3,-1)$, and $\mathbf{x}_3 = (-3,7,1,3)$.

First we apply the Gram-Schmidt process to vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and obtain an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ for the subspace V. Next we compute the orthogonal projection $\mathbf{p}$ of the vector $\mathbf{y}$ onto V:

$$\mathbf{p} = \frac{\langle \mathbf{y}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{y}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \frac{\langle \mathbf{y}, \mathbf{v}_3 \rangle}{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle}\mathbf{v}_3.$$

Then the distance from $\mathbf{y}$ to V equals $\|\mathbf{y} - \mathbf{p}\|$.

Alternatively, we can apply the Gram-Schmidt process to vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}$. We should obtain an orthogonal system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Then the desired distance will be $\|\mathbf{v}_4\|$.

$$\mathbf{x}_1 = (1, -1, 1, -1),\ \mathbf{x}_2 = (1, 1, 3, -1),$$
$$\mathbf{x}_3 = (-3, 7, 1, 3),\ \mathbf{y} = (0, 0, 0, 1).$$

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, -1, 1, -1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = (1, 1, 3, -1) - \frac{4}{4}(1, -1, 1, -1)$$

$$= (0, 2, 2, 0),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

$$= (-3, 7, 1, 3) - \frac{-12}{4}(1, -1, 1, -1) - \frac{16}{8}(0, 2, 2, 0)$$

$$= (0, 0, 0, 0).$$

The Gram-Schmidt process can be used to check linear independence of vectors! It failed because the vector $\mathbf{x}_3$ is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. V is a plane, not a 3-dimensional subspace. To fix things, it is enough to drop $\mathbf{x}_3$, i.e., we should orthogonalize vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}$.

$$\tilde{\mathbf{v}}_3 = \mathbf{y} - \frac{\langle \mathbf{y}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{y}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

$$= (0,0,0,1) - \frac{-1}{4}(1,-1,1,-1) - \frac{0}{8}(0,2,2,0)$$

$$= (1/4, -1/4, 1/4, 3/4).$$

$$\|\tilde{\mathbf{v}}_3\| = \left|\left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right)\right| = \frac{1}{4}\left|(1,-1,1,3)\right| = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}.$$

Problem. The base of a pyramid is a quadrilateral with vertices at points $(0,0,0)$, $(-1,1,2)$, $(1,1,0)$ and $(1,3,2)$. The apex is at the point $(1,0,3)$. Find the volume of the pyramid.

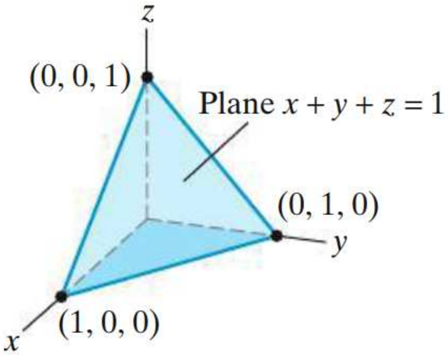
Let P denote the pyramid. Let $O=(0,0,0)$, $A_1=(-1,1,2)$, $A_2=(1,1,0)$, $A_3=(1,3,2)$ and $B=(1,0,3)$.

First we construct a linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ such that $T(1,0,0)=A_1$, $T(0,1,0)=A_2$ and $T(0,0,1)=B$. This transformation is unique and given by

$$T\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix} -1 & 1 & 1\\ 1 & 1 & 0\\ 2 & 0 & 3\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix}.$$

The matrix is $M=(A_1,A_2,B)$.

$$\det M=\begin{vmatrix} -1 & 1 & 1\\ 1 & 1 & 0\\ 2 & 0 & 3\end{vmatrix}=\begin{vmatrix} -2 & 0 & 1\\ 1 & 1 & 0\\ 2 & 0 & 3\end{vmatrix}=\begin{vmatrix} -2 & 1\\ 2 & 3\end{vmatrix}=-8.$$


z
(0, 0, 1)
Plane $x + y + z = 1$
(0, 1, 0)
y
x
(1, 0, 0)

By construction, $T^{-1}(P)$ is a pyramid with the apex at $(0,0,1)$ and three vertices of the base at $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$. It follows that the base of $T^{-1}(P)$ is contained in the xy-plane and that the edge $(0,0,0)$–$(0,0,1)$ is the altitude.

To find the remaining vertex $T^{-1}(A_3)$, we need to solve a system of linear equations:

$$\begin{cases} -x+y+z=1, \\ x+y=3, \\ 2x+3z=2 \end{cases} \iff \begin{cases} x=1, \\ y=2, \\ z=0. \end{cases}$$

Hence the base of the pyramid is a trapezoid with bases of length 1 and 2, and height 1. Its area equals $\frac{3}{2}$. Therefore the volume of the pyramid $T^{-1}(P)$ equals $\frac{1}{3}\cdot\frac{3}{2}\cdot 1=\frac{1}{2}$.

We have $\mathrm{volume}(T(D)) = |\det M|\,\mathrm{volume}(D)$ for any domain $D \subset \mathbb{R}^3$. In particular, $\mathrm{volume}(P) = |-8|\cdot\frac{1}{2}=4$.