

MATH 323
Linear Algebra

**Lecture 8a:
Determinants (continued).**

More properties of determinants

Determinants and matrix multiplication:

- if A and B are $n \times n$ matrices then
$$\det(AB) = \det A \cdot \det B,$$
- if A and B are $n \times n$ matrices then
$$\det(AB) = \det(BA),$$
- if A is an invertible matrix then
$$\det(A^{-1}) = (\det A)^{-1}.$$

Determinants and scalar multiplication:

- if A is an $n \times n$ matrix and $r \in \mathbb{R}$ then
$$\det(rA) = r^n \det A.$$

Examples

$$X = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -2 & 1 \end{pmatrix}.$$

$$\det X = (-1) \cdot 2 \cdot (-3) = 6, \quad \det Y = \det Y^T = 3,$$

$$\det(XY) = 6 \cdot 3 = 18, \quad \det(YX) = 3 \cdot 6 = 18,$$

$$\det(Y^{-1}) = 1/3, \quad \det(XY^{-1}) = 6/3 = 2,$$

$$\det(XYX^{-1}) = \det Y = 3, \quad \det(X^{-1}Y^{-1}XY) = 1,$$

$$\det(2X) = 2^3 \det X = 2^3 \cdot 6 = 48,$$

$$\det(-3X^TXY^{-4}) = (-3)^3 \cdot 6 \cdot 6 \cdot 3^{-4} = -12.$$

Row and column expansions

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the i th row and the j th column of A .

Theorem For any $1 \leq k, m \leq n$ we have that

$$\det A = \sum_{j=1}^n (-1)^{k+j} a_{kj} \det M_{kj},$$

(expansion by k th row)

$$\det A = \sum_{i=1}^n (-1)^{i+m} a_{im} \det M_{im}.$$

(expansion by m th column)

Determinants and the inverse matrix

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the i th row and the j th column of A . The **cofactor matrix** of A is an $n \times n$ matrix $\tilde{A} = (\alpha_{ij})$ defined by $\alpha_{ij} = (-1)^{i+j} \det M_{ij}$.

Theorem $\tilde{A}^T A = A \tilde{A}^T = (\det A)I$.

Sketch of the proof: $A \tilde{A}^T = (\det A)I$ means that

$$\sum_{j=1}^n (-1)^{k+j} a_{kj} \det M_{kj} = \det A \quad \text{for all } k,$$

$$\sum_{j=1}^n (-1)^{k+j} a_{mj} \det M_{kj} = 0 \quad \text{for } m \neq k.$$

Indeed, the 1st equality is the expansion of $\det A$ by the k th row. The 2nd equality is an analogous expansion of $\det B$, where the matrix B is obtained from A by replacing its k th row with a copy of the m th row (clearly, $\det B = 0$).

$\tilde{A}^T A = (\det A)I$ is verified similarly, using column expansions.

Corollary If $\det A \neq 0$ then $A^{-1} = (\det A)^{-1} \tilde{A}^T$.