

MATH 323

Linear Algebra

**Lecture 10:**

**Span. Spanning set.**

**Linear independence.**

## Subspaces of vector spaces

*Definition.* A vector space  $V_0$  is a **subspace** of a vector space  $V$  if  $V_0 \subset V$  and the linear operations on  $V_0$  agree with the linear operations on  $V$ .

**Proposition** A subset  $S$  of a vector space  $V$  is a subspace of  $V$  if and only if  $S$  is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

*Remarks.* The zero vector in a subspace is the same as the zero vector in  $V$ . Also, the subtraction in a subspace agrees with that in  $V$ .

Let  $V$  be a vector space and  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$ .  
Consider the set  $L$  of all linear combinations  
 $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n$ , where  $r_1, r_2, \dots, r_n \in \mathbb{R}$ .

**Theorem**  $L$  is a subspace of  $V$ .

*Proof:* First of all,  $L$  is not empty. For example,  
 $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_n$  belongs to  $L$ .

The set  $L$  is closed under addition since

$$\begin{aligned}(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n) + (s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \dots + s_n\mathbf{v}_n) &= \\ &= (r_1 + s_1)\mathbf{v}_1 + (r_2 + s_2)\mathbf{v}_2 + \dots + (r_n + s_n)\mathbf{v}_n.\end{aligned}$$

The set  $L$  is closed under scalar multiplication since

$$t(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n) = (tr_1)\mathbf{v}_1 + (tr_2)\mathbf{v}_2 + \dots + (tr_n)\mathbf{v}_n.$$

Thus  $L$  is a subspace of  $V$ .

## Span: implicit definition

Let  $S$  be a subset of a vector space  $V$ .

*Definition.* The **span** of the set  $S$ , denoted  $\text{Span}(S)$ , is the smallest subspace of  $V$  that contains  $S$ . That is,

- $\text{Span}(S)$  is a subspace of  $V$ ;
- for any subspace  $W \subset V$  one has
$$S \subset W \implies \text{Span}(S) \subset W.$$

*Remark.* The span of any set  $S \subset V$  is well defined (namely, it is the intersection of all subspaces of  $V$  that contain  $S$ ).

## Span: effective description

Let  $S$  be a subset of a vector space  $V$ .

- If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  then  $\text{Span}(S)$  is the set of all linear combinations  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_n\mathbf{v}_n$ , where  $r_1, r_2, \dots, r_n \in \mathbb{R}$ .
- If  $S$  is an infinite set then  $\text{Span}(S)$  is the set of all linear combinations  $r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \dots + r_k\mathbf{u}_k$ , where  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in S$  and  $r_1, r_2, \dots, r_k \in \mathbb{R}$  ( $k \geq 1$ ).
- If  $S$  is the empty set then  $\text{Span}(S) = \{\mathbf{0}\}$ .

*Examples of subspaces of  $\mathcal{M}_{2,2}(\mathbb{R})$ :*

- The span of  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  consists of all matrices of the form

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

This is the subspace of diagonal matrices.

- The span of  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  consists of all matrices of the form

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

This is the subspace of symmetric matrices ( $A^T = A$ ).

*Examples of subspaces of  $\mathcal{M}_{2,2}(\mathbb{R})$ :*

- The span of  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  is the subspace of anti-symmetric matrices ( $A^T = -A$ ).
- The span of  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , and  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  is the subspace of upper triangular matrices.
- The span of  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  is the entire space  $\mathcal{M}_{2,2}(\mathbb{R})$ .

## Spanning set

*Definition.* A subset  $S$  of a vector space  $V$  is called a **spanning set** for  $V$  if  $\text{Span}(S) = V$ .

*Examples.*

- Vectors  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$ , and  $\mathbf{e}_3 = (0, 0, 1)$  form a spanning set for  $\mathbb{R}^3$  as

$$(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

- Matrices  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a spanning set for  $\mathcal{M}_{2,2}(\mathbb{R})$  as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$



**Problem** Let  $\mathbf{v}_1 = (1, 2, 0)$ ,  $\mathbf{v}_2 = (3, 1, 1)$ , and  $\mathbf{w} = (4, -7, 3)$ . Determine whether  $\mathbf{w}$  belongs to  $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ .

We have to check if there exist  $r_1, r_2 \in \mathbb{R}$  such that  $\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2$ . This vector equation is equivalent to a system of linear equations:

$$\begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix} = r_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + r_2 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \iff \begin{cases} 4 = r_1 + 3r_2 \\ -7 = 2r_1 + r_2 \\ 3 = 0r_1 + r_2 \end{cases}$$

The system has a unique solution:  $r_1 = -5$ ,  $r_2 = 3$ . Thus  $\mathbf{w} = -5\mathbf{v}_1 + 3\mathbf{v}_2$  is in  $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ .

**Problem** Let  $\mathbf{v}_1 = (2, 5)$  and  $\mathbf{v}_2 = (1, 3)$ . Show that  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a spanning set for  $\mathbb{R}^2$ .

Take any vector  $\mathbf{w} = (a, b) \in \mathbb{R}^2$ . We have to check that there exist  $r_1, r_2 \in \mathbb{R}$  such that

$$\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = a \\ 5r_1 + 3r_2 = b \end{cases}$$

Coefficient matrix:  $C = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ .  $\det C = 1 \neq 0$ .

Since the matrix  $C$  is invertible, the system has a unique solution for any  $a$  and  $b$ .

Thus  $\text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2$ .

**Problem** Let  $\mathbf{v}_1 = (2, 5)$  and  $\mathbf{v}_2 = (1, 3)$ . Show that  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a spanning set for  $\mathbb{R}^2$ .

*Alternative solution:* First let us show that vectors  $\mathbf{e}_1 = (1, 0)$  and  $\mathbf{e}_2 = (0, 1)$  belong to  $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ .

$$\mathbf{e}_1 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 1 \\ 5r_1 + 3r_2 = 0 \end{cases} \iff \begin{cases} r_1 = 3 \\ r_2 = -5 \end{cases}$$

$$\mathbf{e}_2 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 0 \\ 5r_1 + 3r_2 = 1 \end{cases} \iff \begin{cases} r_1 = -1 \\ r_2 = 2 \end{cases}$$

Thus  $\mathbf{e}_1 = 3\mathbf{v}_1 - 5\mathbf{v}_2$  and  $\mathbf{e}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2$ .

Then for any vector  $\mathbf{w} = (a, b) \in \mathbb{R}^2$  we have

$$\begin{aligned} \mathbf{w} &= a\mathbf{e}_1 + b\mathbf{e}_2 = a(3\mathbf{v}_1 - 5\mathbf{v}_2) + b(-\mathbf{v}_1 + 2\mathbf{v}_2) \\ &= (3a - b)\mathbf{v}_1 + (-5a + 2b)\mathbf{v}_2. \end{aligned}$$

**Problem** Let  $\mathbf{v}_1 = (2, 5)$  and  $\mathbf{v}_2 = (1, 3)$ . Show that  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a spanning set for  $\mathbb{R}^2$ .

*Remarks on the alternative solution:*

Notice that  $\mathbb{R}^2$  is spanned by vectors  $\mathbf{e}_1 = (1, 0)$  and  $\mathbf{e}_2 = (0, 1)$  since  $(a, b) = a\mathbf{e}_1 + b\mathbf{e}_2$ .

This is why we have checked that vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$  belong to  $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ . Then

$$\begin{aligned} \mathbf{e}_1, \mathbf{e}_2 \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2) &\implies \text{Span}(\mathbf{e}_1, \mathbf{e}_2) \subset \text{Span}(\mathbf{v}_1, \mathbf{v}_2) \\ &\implies \mathbb{R}^2 \subset \text{Span}(\mathbf{v}_1, \mathbf{v}_2) \implies \text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2. \end{aligned}$$

In general, to show that  $\text{Span}(S_1) = \text{Span}(S_2)$ , it is enough to check that  $S_1 \subset \text{Span}(S_2)$  and  $S_2 \subset \text{Span}(S_1)$ .

## More properties of span

Let  $S_0$  and  $S$  be subsets of a vector space  $V$ .

- $S_0 \subset S \implies \text{Span}(S_0) \subset \text{Span}(S)$ .
- $\text{Span}(S_0) = V$  and  $S_0 \subset S \implies \text{Span}(S) = V$ .
- If  $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$  is a spanning set for  $V$  and  $\mathbf{v}_0$  is a linear combination of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  then  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is also a spanning set for  $V$ .

Indeed, if  $\mathbf{v}_0 = r_1\mathbf{v}_1 + \dots + r_k\mathbf{v}_k$ , then

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k = (t_0r_1 + t_1)\mathbf{v}_1 + \dots + (t_0r_k + t_k)\mathbf{v}_k.$$

- $\text{Span}(S_0 \cup \{\mathbf{v}_0\}) = \text{Span}(S_0)$  if and only if  $\mathbf{v}_0 \in \text{Span}(S_0)$ .

If  $\mathbf{v}_0 \in \text{Span}(S_0)$ , then  $S_0 \cup \{\mathbf{v}_0\} \subset \text{Span}(S_0)$ , which implies  $\text{Span}(S_0 \cup \{\mathbf{v}_0\}) \subset \text{Span}(S_0)$ . On the other hand,  $\text{Span}(S_0) \subset \text{Span}(S_0 \cup \{\mathbf{v}_0\})$ .

## Linear independence

*Definition.* Let  $V$  be a vector space. Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$  are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients  $r_1, \dots, r_k \in \mathbb{R}$  are not all equal to zero. Otherwise vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \dots = r_k = 0.$$

A set  $S \subset V$  is **linearly dependent** if one can find some distinct linearly dependent vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  in  $S$ . Otherwise  $S$  is **linearly independent**.

## Examples of linear independence

- Vectors  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$ , and  $\mathbf{e}_3 = (0, 0, 1)$  in  $\mathbb{R}^3$ .

$$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = \mathbf{0} \implies (x, y, z) = \mathbf{0} \\ \implies x = y = z = 0$$

- Matrices  $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$aE_{11} + bE_{12} + cE_{21} + dE_{22} = O \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O \\ \implies a = b = c = d = 0$$

## Examples of linear independence

- Polynomials  $1, x, x^2, \dots, x^n$ .

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0 \text{ identically}$$

$$\implies a_i = 0 \text{ for } 0 \leq i \leq n$$

- The infinite set  $\{1, x, x^2, \dots, x^n, \dots\}$ .

- Polynomials  $p_1(x) = 1$ ,  $p_2(x) = x - 1$ , and  $p_3(x) = (x - 1)^2$ .

$$\begin{aligned} a_1p_1(x) + a_2p_2(x) + a_3p_3(x) &= a_1 + a_2(x - 1) + a_3(x - 1)^2 = \\ &= (a_1 - a_2 + a_3) + (a_2 - 2a_3)x + a_3x^2. \end{aligned}$$

$$\text{Hence } a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = 0 \text{ identically}$$

$$\implies a_1 - a_2 + a_3 = a_2 - 2a_3 = a_3 = 0$$

$$\implies a_1 = a_2 = a_3 = 0$$



**Problem** Let  $\mathbf{v}_1 = (1, 2, 0)$ ,  $\mathbf{v}_2 = (3, 1, 1)$ , and  $\mathbf{v}_3 = (4, -7, 3)$ . Determine whether vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent.

We have to check if there exist  $r_1, r_2, r_3 \in \mathbb{R}$  not all zero such that  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 = \mathbf{0}$ .

This vector equation is equivalent to a system

$$\begin{cases} r_1 + 3r_2 + 4r_3 = 0 \\ 2r_1 + r_2 - 7r_3 = 0 \\ 0r_1 + r_2 + 3r_3 = 0 \end{cases} \quad \left( \begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 1 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array} \right)$$

The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly dependent if and only if the coefficient matrix  $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  is singular. We obtain that  $\det A = 0$  (it is singular).