MATH 323

Linear Algebra

Lecture 13:
Review for Test 1.

Topics for Test 1

Part I: Elementary linear algebra (Leon/de Pillis 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for Test 1

Part II: Abstract linear algebra (Leon/de Pillis 3.1–3.4)

- Definition of a vector space.
- Basic examples of vector spaces.
- Basic properties of vector spaces.
- Subspaces of vector spaces.
- Span, spanning set.
- Linear independence.
- Basis and dimension.

Sample problems for Test 1

Problem 1 Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Problem 2 Let A be a square matrix such that $A^3 = O$.
(i) Prove that the matrix A is not invertible.
(ii) Prove that the matrix $A + I$ is invertible.

Problem 3 Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.
(ii) Find the inverse matrix A^{-1}.

Sample problems for Test 1

Problem 4 Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

Problem 5 Determine which of the following subsets of $\mathbb{R}^\infty$ are subspaces. Briefly explain.

(i) The set S_1 of all arithmetic progressions.
(ii) The set S_2 of all geometric progressions.
(iii) The set S_3 of all square-summable sequences, i.e., sequences $(x_1, x_2, x_3, \dots)$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$.

Sample problems for Test 1

Problem 6 Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Problem 7 Let V denote the solution set of a system

$$\begin{cases} x_2 + 2x_3 + 3x_4 = 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0. \end{cases}$$

Find a basis for this subspace of $\mathbb{R}^4$, then extend it to a basis for $\mathbb{R}^4$.

Problem 1. Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Let $p(x) = a + bx + cx^2$. Then $p(1) = a + b + c$, $p(2) = a + 2b + 4c$, and $p(3) = a + 3b + 9c$.

The coefficients a, b, and c have to be chosen so that

$$\begin{cases} a + b + c = 1, \\ a + 2b + 4c = 3, \\ a + 3b + 9c = 7. \end{cases}$$

We solve this system of linear equations using elementary operations:

$$\begin{cases} a + b + c = 1 \\ a + 2b + 4c = 3 \\ a + 3b + 9c = 7 \end{cases} \iff \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ a + 3b + 9c = 7 \end{cases}$$

$$\iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ a+3b+9c=7 \end{cases} \iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ 2b+8c=6 \end{cases}$$

$$\iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ b+4c=3 \end{cases} \iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ c=1 \end{cases}$$

$$\iff \begin{cases} a+b+c=1 \\ b=-1 \\ c=1 \end{cases} \iff \begin{cases} a=1 \\ b=-1 \\ c=1 \end{cases}$$

Thus the desired polynomial is $p(x) = x^2 - x + 1$.

Problem 2. Let A be a square matrix such that $A^3 = O$.

(i) Prove that the matrix A is not invertible.

The proof is by contradiction. Assume that A is invertible. Since any product of invertible matrices is also invertible, the matrix $A^3 = AAA$ should be invertible as well. However $A^3 = O$ is singular.

Problem 2. Let A be a square matrix such that $A^3 = O$.

(ii) Prove that the matrix $A + I$ is invertible.

It is enough to show that the equation $(A + I)\mathbf{x} = \mathbf{0}$ (where $\mathbf{x}$ and $\mathbf{0}$ are column vectors) has a unique solution $\mathbf{x} = \mathbf{0}$.
Indeed, $(A + I)\mathbf{x} = \mathbf{0} \implies A\mathbf{x} + I\mathbf{x} = \mathbf{0} \implies A\mathbf{x} = -\mathbf{x}$.
Then $A^2\mathbf{x} = A(A\mathbf{x}) = A(-\mathbf{x}) = -A\mathbf{x} = -(-\mathbf{x}) = \mathbf{x}$.
Further, $A^3\mathbf{x} = A(A^2\mathbf{x}) = A\mathbf{x} = -\mathbf{x}$. On the other hand, $A^3\mathbf{x} = O\mathbf{x} = \mathbf{0}$. Hence $-\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$.

Alternatively, we can use equalities

$$X^3 + Y^3 = (X+Y)(X^2 - XY + Y^2) = (X^2 - XY + Y^2)(X+Y),$$

which hold whenever matrices X and Y commute: $XY = YX$.
In particular, they hold for $X = A$ and $Y = I$. We obtain

$$(A + I)(A^2 - A + I) = (A^2 - A + I)(A + I) = A^3 + I^3 = I$$

so that $(A + I)^{-1} = A^2 - A + I$.

Problem 3. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

Subtract the 4th row of A from the 3rd row:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 3rd row:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 3rd column:

$$(-1) \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix} = (-1) \left(\begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} \right) = -1.$$

Problem 3. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(ii) Find the inverse matrix A^{-1}.

First we merge the matrix A with the identity matrix into one 4×8 matrix

$$(A \,|\, I) = \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract 2 times the 1st row from the 2nd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 1st row from the 3rd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 1st row from the 4th row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 4th row from the 2nd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Subtract the 4th row from the 3rd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Add 4 times the 2nd row to the 4th row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array}\right)$$

Add 32 times the 3rd row to the 4th row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Multiply the 2nd, the 3rd, and the 4th rows by -1:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -10 & 0 & -2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right)$$

Subtract the 4th row from the 1st row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & 1 & -10 & 0 & -2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right)$$

Add 10 times the 3rd row to the 2nd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right)$$

Subtract 4 times the 3rd row from the 1st row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right)$$

Add 2 times the 2nd row to the 1st row:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right) = (I \mid A^{-1})$$

Finally the left part of our 4×8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}.$$

Problem 3. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

Alternative solution: We have transformed A into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by -1.

It follows that $\det I = (-1)^3 \det A$.
$\implies \det A = -\det I = -1$.

Problem 4. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.

$(0,0,0) \in S_1 \implies S_1$ is not empty.

$xyz = 0 \implies (rx)(ry)(rz) = r^3 xyz = 0.$
That is, $\mathbf{v} = (x, y, z) \in S_1 \implies r\mathbf{v} = (rx, ry, rz) \in S_1$.
Hence S_1 is closed under scalar multiplication.

However S_1 is not closed under addition.
Counterexample: $(1,1,0) + (0,0,1) = (1,1,1)$.

Problem 4. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.

$(0, 0, 0) \in S_2 \implies S_2$ is not empty.

$x + y + z = 0 \implies rx + ry + rz = r(x + y + z) = 0.$
Hence S_2 is closed under scalar multiplication.

$x + y + z = x' + y' + z' = 0 \implies$
$(x + x') + (y + y') + (z + z') = (x + y + z) + (x' + y' + z') = 0.$
That is, $\mathbf{v} = (x, y, z),\ \mathbf{v}' = (x', y', z') \in S_2$
$\implies \mathbf{v} + \mathbf{v}' = (x + x', y + y', z + z') \in S_2.$
Hence S_2 is closed under addition.

(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.

$y^2 + z^2 = 0 \iff y = z = 0.$

Now it is easy to see that S_3 is a nonempty set closed under addition and scalar multiplication. Alternatively, S_3 is the solution set of a system of linear homogeneous equations, hence a subspace.

(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

S_4 is a nonempty set closed under scalar multiplication.
However S_4 is not closed under addition.
Counterexample: $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$.

Problem 5. Determine which of the following subsets of $\mathbb{R}^\infty$ are subspaces. Briefly explain.

(i) S_1: arithmetic progressions.

A sequence $\mathbf{x} = (x_1, x_2, x_3, \dots)$ is an arithmetic progression if $x_{n+1} = x_n + d$ for some $d \in \mathbb{R}$ and all n.

$\mathbf{0} = (0, 0, 0, \dots)$ is an arithmetic progression with common difference $d = 0$. Hence $\mathbf{0} \in S_1 \implies S_1$ is not empty.

Suppose $\mathbf{x} = (x_1, x_2, x_3, \dots)$ and $\mathbf{y} = (y_1, y_2, y_3, \dots)$ are arithmetic progressions. That is, $x_{n+1} = x_n + d$ and $y_{n+1} = y_n + d'$ for some $d, d' \in \mathbb{R}$ and all n. Then $x_{n+1} + y_{n+1} = (x_n + d) + (y_n + d') = (x_n + y_n) + (d + d')$ for all n so that $\mathbf{x} + \mathbf{y}$ is an arithmetic progression with common difference $d + d'$. Also, $rx_{n+1} = rx_n + rd$ for any scalar r and all n. Hence $r\mathbf{x}$ is an arithmetic progression with common difference rd.

Therefore the set S_1 is closed under addition and scalar multiplication. Thus S_1 is a subspace of $\mathbb{R}^\infty$.

Problem 5. Determine which of the following subsets of $\mathbb{R}^\infty$ are subspaces. Briefly explain.

(ii) S_2: geometric progressions.

A sequence $\mathbf{x} = (x_1, x_2, x_3, \dots)$ is a geometric progression if $x_{n+1} = x_n q$ for some $q \neq 0$ and all n.

$\mathbf{0} = (0, 0, 0, \dots)$ is a geometric progression with common ratio $q = 1$. Hence $\mathbf{0} \in S_2 \implies S_2$ is not empty.

Suppose $\mathbf{x} = (x_1, x_2, x_3, \dots)$ is a geometric progression with common ratio q. Then $rx_{n+1} = r(x_n q) = (rx_n)q$ for any scalar r and all n. Hence $r\mathbf{x}$ is also a geometric progression with the same common ratio q. Therefore the set S_2 is closed under scalar multiplication.

However S_2 is not closed under addition. Counterexample:
$(1, 1, 1, \dots) + (2, 4, 8, \dots, 2^n, \dots) = (3, 5, 9, \dots, 2^n{+}1, \dots)$.

Thus S_2 is not a subspace of $\mathbb{R}^\infty$.

Problem 5. Determine which of the following subsets of $\mathbb{R}^\infty$ are subspaces. Briefly explain.

(iii) S_3: square-summable sequences.

A sequence $\mathbf{x} = (x_1, x_2, x_3, \dots)$ is called square-summable if the series $\sum_{n=1}^{\infty} |x_n|^2$ converges.

For $\mathbf{0} = (0, 0, 0, \dots)$, we have $\sum_{n=1}^{\infty} |0|^2 = 0 < \infty$. Hence $\mathbf{0} \in S_3 \implies S_3$ is not empty.

Suppose $\mathbf{x} = (x_1, x_2, x_3, \dots)$ and $\mathbf{y} = (y_1, y_2, y_3, \dots)$ are both square-summable. Using the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we obtain $|x_n + y_n|^2 \le 2|x_n|^2 + 2|y_n|^2$ for all n. Hence

$$\sum_{n=1}^{\infty} |x_n + y_n|^2 \le 2\sum_{n=1}^{\infty} |x_n|^2 + 2\sum_{n=1}^{\infty} |y_n|^2 < \infty$$

so that $\mathbf{x} + \mathbf{y} \in S_3$. Also, $\sum_{n=1}^{\infty} |rx_n|^2 = |r|^2 \sum_{n=1}^{\infty} |x_n|^2 < \infty$ for any scalar r so that $r\mathbf{x} \in S_3$.

Therefore the set S_3 is closed under addition and scalar multiplication. Thus S_3 is a subspace of $\mathbb{R}^\infty$.

Problem 7. Let V denote the solution set of a system

$$\begin{cases} x_2 + 2x_3 + 3x_4 = 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0. \end{cases}$$

Find a basis for this subspace of $\mathbb{R}^4$.

To find a basis, we need to solve the system. To this end, we subtract 2 times the 1st equation from the 2nd one, then switch the equations:

$$\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{cases}$$

x_3 and x_4 are free variables. General solution:

$$\begin{cases} x_1 = t + 2s \\ x_2 = -2t - 3s \\ x_3 = t \\ x_4 = s \end{cases} \qquad (t, s \in \mathbb{R})$$

In vector form, $(x_1, x_2, x_3, x_4) = t(1, -2, 1, 0) + s(2, -3, 0, 1)$. Hence vectors $(1, -2, 1, 0)$ and $(2, -3, 0, 1)$ span V. Since they are also linearly independent, they form a basis for V.

Problem 7. Let V denote the solution set of a system

$$\begin{cases} x_2 + 2x_3 + 3x_4 = 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0. \end{cases}$$

Find a basis for this subspace of $\mathbb{R}^4$, then extend it to a basis for $\mathbb{R}^4$.

Vectors $\mathbf{v}_1 = (1, -2, 1, 0)$ and $\mathbf{v}_2 = (2, -3, 0, 1)$ form a basis for V. To extend them to a basis for $\mathbb{R}^4$, we need to add two vectors $\mathbf{v}_3$ and $\mathbf{v}_4$ so that four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent. We can choose new vectors from the standard basis (or any other spanning set for $\mathbb{R}^4$). For example, we can add $\mathbf{e}_1 = (1, 0, 0, 0)$ and $\mathbf{e}_2 = (0, 1, 0, 0)$. To verify linear independence of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2$, we check that the matrix whose columns are these vectors is invertible. Indeed,

$$\begin{vmatrix} 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Problem 6. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$ and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

The functions f_1, f_2, f_3 are linearly independent whenever the Wronskian $W[f_1, f_2, f_3]$ is not identically zero.

$$W[f_1, f_2, f_3](x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} = \begin{vmatrix} x & xe^x & e^{-x} \\ 1 & e^x + xe^x & -e^{-x} \\ 0 & 2e^x + xe^x & e^{-x} \end{vmatrix}$$

$$= e^{-x} \begin{vmatrix} x & xe^x & 1 \\ 1 & e^x + xe^x & -1 \\ 0 & 2e^x + xe^x & 1 \end{vmatrix} = \begin{vmatrix} x & x & 1 \\ 1 & 1+x & -1 \\ 0 & 2+x & 1 \end{vmatrix}$$

$$= x \begin{vmatrix} 1+x & -1 \\ 2+x & 1 \end{vmatrix} - \begin{vmatrix} x & 1 \\ 2+x & 1 \end{vmatrix} = x(2x+3) + 2 = 2x^2 + 3x + 2.$$

The polynomial $2x^2 + 3x + 2$ is never zero.

Problem 6. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$ and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Alternative solution: Suppose that $af_1(x)+bf_2(x)+cf_3(x) = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Let us differentiate this identity:

$$ax + bxe^x + ce^{-x} = 0,$$
$$a + be^x + bxe^x - ce^{-x} = 0,$$
$$2be^x + bxe^x + ce^{-x} = 0,$$
$$3be^x + bxe^x - ce^{-x} = 0,$$
$$4be^x + bxe^x + ce^{-x} = 0.$$

(the 5th identity)−(the 3rd identity): $2be^x = 0 \implies b = 0.$

Substitute $b = 0$ in the 3rd identity: $ce^{-x} = 0 \implies c = 0.$

Substitute $b = c = 0$ in the 2nd identity: $a = 0.$

Problem 6. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$ and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Alternative solution: Suppose that $ax + bxe^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

For any $x \neq 0$ divide both sides of the identity by xe^x:

$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$

The left-hand side approaches b as $x \to +\infty$. $\implies b = 0$

Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the identity by x:

$$a + cx^{-1}e^{-x} = 0.$$

The left-hand side approaches a as $x \to +\infty$. $\implies a = 0$

Now $ce^{-x} = 0 \implies c = 0$.