MATH 323 Fall 2023

Sample problems for Test 2

(to be worked out during the review)

Problem 1 Let $A = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(i) Find the rank and the nullity of the matrix A.
(ii) Find a basis for the row space of A, then extend this basis to a basis for $\mathbb{R}^4$.
(iii) Find a basis for the nullspace of A.

Problem 2 Let A and B be two matrices such that the product AB is well defined.
(i) Prove that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.
(ii) Prove that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

Problem 3 Complex numbers $\mathbb{C}$ form a vector space of (real) dimension 2. Consider a function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = (3 + 2i)z$ for all $z \in \mathbb{C}$.
(i) Prove that f is a linear operator on the vector space $\mathbb{C}$.
(ii) Find the matrix of f relative to the basis 1, i.

Problem 4 Let V be a subspace of $\mathcal{F}(\mathbb{R})$ spanned by functions e^x and e^{-x}. Let L be a linear operator on V such that

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

is the matrix of L relative to the basis e^x, e^{-x}. Find the matrix of L relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Problem 5 Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.
(ii) For each eigenvalue of A, find an associated eigenvector.
(iii) Is the matrix A diagonalizable? Explain.
(iv) Find all eigenvalues of the matrix A^2.

Problem 6 Find a linear polynomial which is the best least squares fit to the following data:

x	-2	-1	0	1	2
$f(x)$	-3	-2	1	2	5

Problem 7 Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.
(i) Find an orthonormal basis for V.
(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.
(iii) Find the distance from the vector $\mathbf{y} = (1, 0, 0, 0)$ to the subspaces V and $V^\perp$.