Challenges

Challenge #1 (50 pts., no deadline)
Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Suppose that for any point $x \in \mathbb{R}$ there exists a derivative of f that vanishes at x:

$$f^{(n)}(x) = 0 \quad \text{for some} \quad n \geq 1.$$

Prove that f is a polynomial.

Remark. A polynomial can be uniquely characterized as an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{(n)}(x) \equiv 0$ (identically zero) for some $n \geq 1$.

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Challenge #2 (5 pts., deadline: September 5)
Construct a strict linear order $\prec$ on the set $\mathbb{C}$ of complex numbers that satisfies Axiom OA:
$a \prec b$ implies $a + c \prec b + c$ for all $a, b, c \in \mathbb{C}$.

Challenge #3 (10 pts., deadline: September 5)
Construct a strict linear order $\prec$ on the set $\mathbb{R}(x)$ of rational functions in variable x with real coefficients that makes $\mathbb{R}(x)$ into an ordered field.

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A set $E \subset \mathbb{R}$ is called an interval if with any two elements it contains all elements of $\mathbb{R}$ that lie between them. To be precise, $a, b \in E$ and $a < c < b$ imply $c \in E$ for all $a, b, c \in \mathbb{R}$.

Challenge #4 (10 pts., deadline: September 12)

Prove the following statements.

(i) If E is a bounded interval that consists of more than one point, then there exist $a, b \in \mathbb{R}$, $a < b$, such that $E = (a, b)$ or $[a, b)$ or $(a, b]$ or $[a, b]$.

(ii) If E is an interval that is neither bounded above nor bounded below, then $E = \mathbb{R}$.

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Challenge #5 (5 pts., deadline: September 19)
Prove that the set $\mathbb{R} \times \mathbb{R}$ is of the same cardinality as $\mathbb{R}$.

Challenge #6 (10 pts., deadline: September 19)
Let $\mathcal{F}(\mathbb{R})$ denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$ and $C(\mathbb{R})$ denote the subset of continuous functions. Prove that the set $C(\mathbb{R})$ is of the same cardinality as $\mathbb{R}$ while the set $\mathcal{F}(\mathbb{R})$ is not.

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Challenge #7 (5 pts., deadline: September 26)

Build a sequence $\{x_n\}$ of real numbers such that every real number is a limit point of $\{x_n\}$ (a limit point of a sequence is, by definition, the limit of a convergent subsequence).

Challenge #8 (10 pts., deadline: September 26)

Let $\{F_n\}$ be the sequence of the Fibonacci numbers: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that (as first observed by Kepler)

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{\sqrt{5}+1}{2}, \text{ the golden ratio.}$$

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Challenge #9 (5 pts., deadline: October 3)
Construct a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at all integer points and discontinuous at all noninteger points.

Challenge #10 (10 pts., deadline: October 3)
Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at all rational points and discontinuous at all irrational points.

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Challenge #11 (10 pts., no deadline)
Prove that the Riemann function

$$R(x) = \begin{cases} 1/q & \text{if } x = p/q, \text{ a reduced fraction, } q > 0, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not differentiable at any point.

Challenge #12 (10 pts., no deadline)
Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(x) > 0$ for all $x \in \mathbb{Q}$ and $f(x) = 0$ for all $x \notin \mathbb{Q}$. Prove that f can not be differentiable at every irrational point.

Challenge #13 (20 pts., no deadline)
Let $R : \mathbb{R} \to \mathbb{R}$ be the Riemann function. Prove that the function R^3 is differentiable at some irrational points.

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Challenge #14 (10 pts., deadline: November 7)

Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in C^\infty(\mathbb{R})$, $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$, $f(x) = 1$ if $|x| \leq 1$, and $f(x) = 0$ if $|x| \geq 2$.

Challenge #15 (10 pts., deadline: November 7)

Suppose that a function $g : \mathbb{R} \to \mathbb{R}$ is locally a polynomial, which means that for every $c \in \mathbb{R}$ there exists $\varepsilon > 0$ such that g coincides with a polynomial on the interval $(c - \varepsilon, c + \varepsilon)$. Prove that g is a polynomial.

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Challenge #16 (10 pts., no deadline)
Let $\{a_n\}$ be a sequence of distinct real numbers converging to a limit b. Suppose that a function f is infinitely differentiable at the point b and $f(a_n) = 0$ for all $n \in \mathbb{N}$. Prove that all derivatives of the function f at b are equal to 0.

Challenge #17 (40 pts., no deadline)
Prove Lebesgue's criterion for Riemann integrability: a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on the interval $[a, b]$ if and only if f is bounded on $[a, b]$ and continuous almost everywhere on $[a, b]$.