MATH 409

Advanced Calculus I

Lecture 1:
Axioms of an ordered field.

Real line

Systematic study of the calculus of functions of one variable begins with the study of the domain of such functions, the set of real numbers $\mathbb{R}$ (real line).

The real line is a mathematical object rich with structure. This includes:

- algebraic structure (4 arithmetic operations);
- ordering (for any three points, one is located between the other two);
- metric structure (we can measure distances between points);
- continuity (we can get from one point to another in a continuous way).

Axiomatic model

The study of the real line begins with formulation of an axiomatic model, which is to provide the solid foundation for all subsequent developments.

The axiomatic model of the real numbers shall be formulated using three **postulates**, each consisting of one or several axioms. To verify that the axiomatic model is adequate, one has to prove that the axioms are **consistent** (namely, there exists an object satisfying them) and **categorical** (namely, that object is, in a sense, unique).

The axioms are chosen among basic properties of the real numbers, which ensures consistency. Postulate 1 formalizes the algebraic structure, Postulate 2 formalizes the ordering, and Postulate 3 formalizes the continuous structure (the metric structure does not require a separate postulate; it can be formalized in terms of the other structures).

Field

The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ motivated the introduction of an abstract algebraic structure called a **field**. Informally, a field is a set with 4 arithmetic operations (addition, subtraction, multiplication, and division) that have roughly the same properties as those of real (or complex) numbers.

The notion of field is important for the linear algebra. Namely, a field is a set that can serve as a set of scalars for a vector space.

Formally, a field is a set F equipped with two binary operations, called **addition** and **multiplication** and denoted accordingly, that satisfy a number of axioms.

Field: formal definition

A **field** is a set F equipped with two operations, **addition** $F \times F \ni (a, b) \mapsto a + b \in F$ and **multiplication** $F \times F \ni (a, b) \mapsto a \cdot b \in F$, such that:

F1. $a + b = b + a$ for all $a, b \in F$.
F2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in F$.
F3. There exists an element of F, denoted 0, such that $a + 0 = 0 + a = a$ for all $a \in F$.
F4. For any $a \in F$ there exists an element of F, denoted $-a$, such that $a + (-a) = (-a) + a = 0$.

F1′. $a \cdot b = b \cdot a$ for all $a, b \in F$.
F2′. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in F$.
F3′. There exists an element of F different from 0, denoted 1, such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in F$.
F4′. For any $a \in F$, $a \neq 0$ there exists an element of F, denoted a^{-1}, such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

F5. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in F$.

Subtraction and **division** in the field F are defined as follows: $a - b = a + (-b)$, $a/b = a \cdot b^{-1}$.

Postulate 1. The set of real numbers $\mathbb{R}$ is a field.

Other examples of fields:

- Complex numbers $\mathbb{C}$.
- Rational numbers $\mathbb{Q}$.
- $\mathbb{R}(X)$: rational functions in variable X with real coefficients.
- $\mathbb{F}_2 = \{\bar{0}, \bar{1}\}$: field of two elements.

The operations are defined as follows:
$\bar{0} + \bar{0} = \bar{1} + \bar{1} = \bar{0}$, $\bar{0} + \bar{1} = \bar{1} + \bar{0} = \bar{1}$,
$\bar{0} \cdot \bar{0} = \bar{0} \cdot \bar{1} = \bar{1} \cdot \bar{0} = \bar{0}$, $\bar{1} \cdot \bar{1} = \bar{1}$.

Basic properties of fields

- The zero 0 is unique.

Suppose z_1 and z_2 are both zeroes, that is,
$a + z_1 = z_1 + a = a$ and $a + z_2 = z_2 + a = a$ for all $a \in F$.
Then $z_1 + z_2 = z_2$ and $z_1 + z_2 = z_1$. Hence $z_1 = z_2$.

- For any $a \in F$, the negative $-a$ is unique.

Suppose b_1 and b_2 are both negatives of a. Let us compute the sum $b_1 + a + b_2$ in two ways:

$$(b_1 + a) + b_2 = 0 + b_2 = b_2,$$
$$b_1 + (a + b_2) = b_1 + 0 = b_1.$$

By associativity of the addition, $b_1 = b_2$.

Basic properties of fields

- (Cancellation law) $a + c = b + c$ implies $a = b$ for any $a, b, c \in F$.

If $a + c = b + c$ then $(a + c) + (-c) = (b + c) + (-c)$. By associativity, $(a + c) + (-c) = a + (c + (-c)) = a + 0 = a$ and $(b + c) + (-c) = b + (c + (-c)) = b + 0 = b$. Hence $a = b$.

- $0 \cdot a = 0$ for any $a \in F$.

We have $0 \cdot a + a = 0 \cdot a + 1 \cdot a = (0 + 1) \cdot a = 1 \cdot a = a$ $= 0 + a$. By the cancellation law, $0 \cdot a = 0$.

- $(-1) \cdot a = -a$ for any $a \in F$.

Indeed, $a + (-1) \cdot a = (-1) \cdot a + a = (-1) \cdot a + 1 \cdot a$ $= (-1 + 1) \cdot a = 0 \cdot a = 0$.

Basic properties of fields

- $-(-a) = a$ for all $a \in F$.
- $(-1) \cdot (-1) = 1$.
- $-(a - b) = b - a$ for all $a, b \in F$.
- The unity 1 is unique.
- For any $a \neq 0$, the inverse a^{-1} is unique.
- (Cancellation law) $ac = bc$ implies $a = b$ whenever $c \neq 0$.
- For any $a, b \in F$ the equality $ab = 0$ implies that $a = 0$ or $b = 0$.

Relations

Recall that the **Cartesian product** $X \times Y$ of two sets X and Y is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$.

The Cartesian square $X \times X$ is also denoted X^2.

Definition. A **relation** R on a set X is (identified with) a subset of its Cartesian square: $R \subset X^2$.

If $(x, y) \in R$, then we say that x **is related to** y (in the sense of R or by R) and write xRy.

Examples of relations

- "is equal to"

$xRy \iff x = y$

- "is not equal to"

$xRy \iff x \neq y$

- "is less than"

$X = \mathbb{R},\ \ xRy \iff x < y$

- "is less than or equal to"

$X = \mathbb{R},\ \ xRy \iff x \leq y$

- "is contained in"

$X =$ the set of all subsets of some set Y,

$xRy \iff x \subset y$

- "divides"

$X = \mathbb{Z},\ \ xRy \iff y/x \in \mathbb{Z}$

Properties of relations

Definition. Let R be a relation on a set X. We say that R is

- **reflexive** if xRx for all $x \in X$,
- **symmetric** if, for all $x, y \in X$, xRy implies yRx,
- **antisymmetric** if, for all $x, y \in X$, xRy and yRx cannot hold simultaneously,
- **weakly antisymmetric** if, for all $x, y \in X$, xRy and yRx imply that $x = y$,
- **transitive** if, for all $x, y, z \in X$, xRy and yRz imply that xRz.

Partial ordering

Definition. A relation R on a set X is a **partial ordering** (or **partial order**, or simply **order**) if R is reflexive, weakly antisymmetric, and transitive:

- xRx,
- xRy and $yRx \implies x = y$,
- xRy and $yRz \implies xRz$.

A relation R on a set X is a **strict partial order** (or **strict order**) if R is antisymmetric and transitive:

- $xRy \implies$ not yRx,
- xRy and $yRz \implies xRz$.

Examples. "is less than or equal to", "is contained in", "is a divisor of" are partial orders. "is less than" is a strict order.

Linear order

Definition. An order R on a set X is called **linear** (or **total**) if for any elements $x, y \in X$ at least one of the following statements hold: xRy, yRx, or $x = y$.

Postulate 2. There is a relation on the set of real numbers $\mathbb{R}$, denoted $<$, that is a strict linear order. Moreover, this order and arithmetic operations on $\mathbb{R}$ satisfy the following axioms:

OA. $a < b$ implies $a + c < b + c$ for all $a, b, c \in \mathbb{R}$.
OM1. $a < b$ and $c > 0$ imply $ac < bc$ for all $a, b, c \in \mathbb{R}$.
OM2. $a < b$ and $c < 0$ imply $bc < ac$ for all $a, b, c \in \mathbb{R}$.

Two axioms OM1 and OM2 can be replaced by one:

OM. $0 < a$ and $0 < b$ imply $0 < ab$ for all $a, b \in \mathbb{R}$.

Auxiliary notation. $a > b$ means that $b < a$. By $a \leq b$ we mean that $a < b$ or $a = b$. By $a < b < c$ we mean that $a < b$ and $b < c$.

Basic properties of linearly ordered fields

- $a > 0$ implies $-a < 0$.
- $a < b$ implies $a - b < 0$.
- $a > 0$ and $b < 0$ imply $ab < 0$.
- $a < 0$ and $b < 0$ imply $ab > 0$.
- $a \neq 0$ implies $a^2 > 0$ (where $a^2 = a \cdot a$).
- $0 < 1$.
- $0 < a < b$ and $0 < c < d$ implies $ac < bd$.
- $0 < a < b$ implies $a^{-1} > b^{-1} > 0$.