MATH 409

Advanced Calculus I

Lecture 5:
Binomial formula.
Inverse function and inverse images.
Countable and uncountable sets.

Well-ordering and induction

Principle of well-ordering:
The set $\mathbb{N}$ is well-ordered, that is, any nonempty subset of $\mathbb{N}$ has a least element.

Principle of mathematical induction:
Let $P(n)$ be an assertion depending on a natural variable n. Suppose that $P(1)$ holds and $P(k)$ implies $P(k+1)$ for any $k \in \mathbb{N}$. Then $P(n)$ holds for all $n \in \mathbb{N}$.

Induction with a different base:
Let $P(n)$ be an assertion depending on an integer variable n. Suppose that $P(n_0)$ holds for some $n_0 \in \mathbb{Z}$ and $P(k)$ implies $P(k+1)$ for any $k \geq n_0$. Then $P(n)$ holds for all $n \geq n_0$.

Strong induction: Let $P(n)$ be an assertion depending on a natural variable n. Suppose that $P(n)$ holds whenever $P(k)$ holds for all natural $k < n$. Then $P(n)$ holds for all $n \in \mathbb{N}$.

Inductive definition

The principle of mathematical induction allows to define mathematical objects inductively (that is, recursively).

Examples of inductive definitions:

- Power a^n of a number

Given a real number a, we let $a^0 = 1$ and $a^n = a^{n-1}a$ for any $n \in \mathbb{N}$.

- Factorial $n!$

We let $0! = 1$ and $n! = (n-1)! \cdot n$ for any $n \in \mathbb{N}$.

- Fibonacci numbers $F_1, F_2, \ldots$

We let $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for any $n \geq 3$.

Binomial coefficients

Definition. For any integers n and k, $0 \le k \le n$, we define the **binomial coefficient** $\binom{n}{k}$ (n choose k) by

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

If $k > 0$ then $\displaystyle\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{1 \cdot 2 \cdot \dots \cdot k}$.

"n choose k" refers to the fact that $\binom{n}{k}$ is the number of all k-element subsets of an n-element set.

Lemma $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ for all n and k, $1 \le k \le n$.

Proof: $\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!}$

$= \frac{n!}{(k-1)!\,(n-k)!}\left(\frac{1}{n-k+1} + \frac{1}{k}\right) = \frac{n!}{(k-1)!\,(n-k)!} \cdot \frac{n+1}{k(n-k+1)}$

$= \frac{(n+1)!}{k!\,(n-k+1)!} = \binom{n+1}{k}$.

Pascal's triangle

The formula $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ allows to compute the binomial coefficients recursively. Usually the results are formatted as a triangular array called **Pascal's triangle**. Namely, $\binom{n}{k}$ is the k-th number in the n-th row of the triangle (the numbering of rows and elements in a row starts from 0).

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

Binomial formula

Theorem For any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

In particular, $(a+b)^2 = a^2 + 2ab + b^2$,
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

The coefficients in the binomial formula are consecutive numbers in the n-th row of Pascal's triangle.

Theorem For any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Proof: The proof is by induction on n. In the case $n = 1$, the formula is trivial: $(a+b)^1 = \binom{1}{0} a + \binom{1}{1} b$. Now assume that the formula holds for a particular value of n. Then

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\ \textstyle\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= \textstyle\sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k \ + \ \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

$$= \textstyle\sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k \ + \ \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n-k+1} b^k$$

$$= \textstyle\binom{n}{0} a^{n+1} \ + \ \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1}\right) a^{n-k+1} b^k \ + \ \binom{n}{n} b^{n+1}$$

$$= \textstyle\binom{n+1}{0} a^{n+1} \ + \ \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k \ + \ \binom{n+1}{n+1} b^{n+1},$$

which completes the induction step.

Functions

A **function** (or **map**) $f : X \to Y$ is an assignment: to each $x \in X$ we assign an element $f(x) \in Y$.

Definition. A function $f : X \to Y$ is **injective** (or **one-to-one**) if $f(x') = f(x) \implies x' = x$.

The function f is **surjective** (or **onto**) if for each $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$.

Finally, f is **bijective** if it is both surjective and injective. Equivalently, if for each $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$.

Suppose we have two functions $f : X \to Y$ and $g : Y \to X$. We say that g is the **inverse function** of f (denoted f^{-1}) if $y = f(x) \iff g(y) = x$ for all $x \in X$ and $y \in Y$.

Theorem The inverse function f^{-1} exists if and only if f is bijective.

Definition. The **composition** of functions $f : X \to Y$ and $g : Y \to Z$ is a function from X to Z, denoted $g \circ f$, that is defined by $(g \circ f)(x) = g(f(x))$, $x \in X$.

Properties of compositions:

- If f and g are one-to-one, then $g \circ f$ is also one-to-one.
- If $g \circ f$ is one-to-one, then f is also one-to-one.
- If f and g are onto, then $g \circ f$ is also onto.
- If $g \circ f$ is onto, then g is also onto.
- If f and g are bijective, then $g \circ f$ is also bijective.
- If f and g are invertible, then $g \circ f$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- If id_Z denotes the identity function on a set Z, then $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for any function $f : X \to Y$.
- For any functions $f : X \to Y$ and $g : Y \to X$, we have $g = f^{-1}$ if and only if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Images and pre-images

Definition. Given a function $f : X \to Y$, the **image** of a set $E \subset X$ under f, denoted $f(E)$, is a subset of Y defined by $f(E) = \{f(x) \mid x \in E\}$. The **pre-image** (or **inverse image**) of a set $D \subset Y$ under f, denoted $f^{-1}(D)$, is a subset of X defined by $f^{-1}(D) = \{x \in X \mid f(x) \in D\}$.

Remark. If the function f is invertible, then the pre-image $f^{-1}(D)$ is also the image of D under the inverse function f^{-1}. However $f^{-1}(D)$ is well defined even if f is not invertible.

Properties of images and pre-images:

- $f\Big(\bigcup_{\alpha \in I} E_\alpha\Big) = \bigcup_{\alpha \in I} f(E_\alpha), \quad f\Big(\bigcap_{\alpha \in I} E_\alpha\Big) \subset \bigcap_{\alpha \in I} f(E_\alpha);$
- $f^{-1}\Big(\bigcup_{\alpha \in I} D_\alpha\Big) = \bigcup_{\alpha \in I} f^{-1}(D_\alpha), \quad f^{-1}\Big(\bigcap_{\alpha \in I} D_\alpha\Big) = \bigcap_{\alpha \in I} f^{-1}(D_\alpha),$
- $f^{-1}(D \setminus D_0) = f^{-1}(D) \setminus f^{-1}(D_0).$

Cardinality of a set

Definition. Given two sets A and B, we say that A is of the same **cardinality** as B if there exists a bijective function $f : A \to B$. Notation: $|A| = |B|$.

Theorem The relation "is of the same cardinality as" is an equivalence relation, i.e., it is reflexive ($|A| = |A|$ for any set A), symmetric ($|A| = |B|$ implies $|B| = |A|$), and transitive ($|A| = |B|$ and $|B| = |C|$ imply $|A| = |C|$).

Proof: The identity map $\mathrm{id}_A : A \to A$ is bijective. If f is a bijection of A onto B, then the inverse map f^{-1} is a bijection of B onto A. If $f : A \to B$ and $g : B \to C$ are bijections then the composition $g \circ f$ is a bijection of A onto C.

Countable and uncountable sets

A nonempty set is **finite** if it is of the same cardinality as $\{1, 2, \ldots, n\} = [1, n] \cap \mathbb{N}$ for some $n \in \mathbb{N}$. Otherwise it is **infinite**.

An infinite set is called **countable** (or **countably infinite**) if it is of the same cardinality as $\mathbb{N}$. Otherwise it is **uncountable** (or **uncountably infinite**).

An infinite set E is countable if it is possible to arrange all elements of E into a single sequence (an infinite list) $x_1, x_2, \ldots$.

Countable sets

Examples of countable sets:

- $\mathbb{N}$: natural numbers
- $2\mathbb{N}$: even natural numbers
- $\mathbb{Z}$: integers
- $\mathbb{N} \times \mathbb{N}$: pairs of natural numbers
- $\mathbb{Q}$: rational numbers
- Algebraic numbers (roots of nonzero polynomials with integer coefficients).

Properties of countable sets:

- Any infinite set contains a countable subset.
- Any infinite subset of a countable set is also countable.
- If $A_1, A_2, \ldots$ are finite or countable sets, then the union $A_1 \cup A_2 \cup \ldots$ is also finite or countable.

Theorem The set $\mathbb{R}$ is uncountable.

Proof: It is enough to prove that the interval $(0, 1)$ is uncountable. Assume the contrary. Then all numbers from $(0, 1)$ can be arranged into an infinite list $x_1, x_2, \dots$ Any number $x \in (0, 1)$ admits a decimal expansion of the form $0.d_1 d_2 d_3 \dots$, where each $d_i \in \{0, 1, \dots, 9\}$. In particular,

$$x_1 = 0.d_{11} d_{12} d_{13} d_{14} d_{15} \dots$$
$$x_2 = 0.d_{21} d_{22} d_{23} d_{24} d_{25} \dots$$
$$x_3 = 0.d_{31} d_{32} d_{33} d_{34} d_{35} \dots$$

. . . .

Now for any $n \in \mathbb{N}$ choose a decimal digit $\tilde{d}_n$ such that $\tilde{d}_n \neq d_{nn}$ and $\tilde{d}_n \notin \{0, 9\}$. Then $0.\tilde{d}_1 \tilde{d}_2 \tilde{d}_3 \dots$ is the decimal expansion of some number $\tilde{x} \in (0, 1)$. By construction, it is different from all expansions in the list. Although some real numbers admit two decimal expansions (e.g., $0.50000\dots$ and $0.49999\dots$), the condition $\tilde{d}_n \notin \{0, 9\}$ ensures that $\tilde{x}$ is not such a number. Thus $\tilde{x}$ is not listed, a contradiction.