MATH 409

Advanced Calculus I

Lecture 8: Monotone sequences (continued). Cauchy sequences. Limit points.

Monotone sequences

Definition. A sequence $\{x_n\}$ is called **increasing** (or **nondecreasing**) if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. It is called **strictly increasing** if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.

Likewise, the sequence $\{x_n\}$ is called **decreasing** (or **nonincreasing**) if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. It is **strictly decreasing** if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.

Increasing and decreasing sequences are called **monotone**.

Theorem Any monotone sequence converges to a limit if bounded, and diverges to infinity otherwise.

Examples

- If $0 < a < 1$ then $a^n \to 0$ as $n \to \infty$.

Since $a < 1$ and $a > 0$, it follows that $a^{n+1} < a^n$ and $a^n > 0$ for all $n \in \mathbb{N}$. Hence the sequence $\{a^n\}$ is strictly decreasing and bounded. Therefore it converges to some $x \in \mathbb{R}$. Since $a^{n+1} = a^n a$ for all n, it follows that $a^{n+1} \to xa$ as $n \to \infty$. However the sequence $\{a^{n+1}\}$ is a subsequence of $\{a^n\}$, hence it converges to the same limit as $\{a^n\}$. Thus $xa = x$, which implies that $x = 0$.

- If $a > 1$ then $a^n \to +\infty$ as $n \to \infty$.

Since $a > 1$, it follows that $a^{n+1} > a^n > 1$ for all $n \in \mathbb{N}$. Hence the sequence $\{a^n\}$ is strictly increasing. Then $\{a^n\}$ either diverges to $+\infty$ or converges to a limit x. In the latter case we argue as above to obtain that $x = 0$. However this contradicts with $a^n > 1$. Thus $\{a^n\}$ diverges to $+\infty$.

Examples

- If $a > 0$ then $\sqrt[n]{a} \to 1$ as $n \to \infty$.

Remark. By definition, $\sqrt[n]{a}$ is a unique positive number r such that $r^n = a$.

If $a \geq 1$ then $a^{n+1} \geq a^n \geq 1$ for all $n \in \mathbb{N}$, which implies that $\sqrt[n(n+1)]{a^{n+1}} \geq \sqrt[n(n+1)]{a^n} \geq 1$. Notice that $\sqrt[n(n+1)]{a^{n+1}} = \sqrt[n]{a}$ and $\sqrt[n(n+1)]{a^n} = \sqrt[n+1]{a}$. Hence $\sqrt[n]{a} \geq \sqrt[n+1]{a} \geq 1$ for all n.
Similarly, in the case $0 < a < 1$ we obtain that $\sqrt[n]{a} < \sqrt[n+1]{a} < 1$ for all n.

In either case, the sequence $\{\sqrt[n]{a}\}$ is monotone and bounded. Therefore it converges to a limit x. Then the sequence $\{\sqrt[2n]{a}\}$ also converges to x since it is a subsequence of $\{\sqrt[n]{a}\}$. At the same time, $(\sqrt[2n]{a})^2 = \sqrt[n]{a}$, which implies that $x^2 = x$. Hence $x = 0$ or $x = 1$. However the limit cannot be 0 since $\sqrt[n]{a} \geq \min(a, 1) > 0$. Thus $x = 1$.

Examples

- The sequence $x_n = \left(1 + \dfrac{1}{n}\right)^n, \; n = 1, 2, 3, \dots,$

is increasing and bounded, hence it is convergent.

Remark. The limit is the number $e = 2.71828\dots$

First let us show that $\{x_n\}$ is increasing. For any $n \in \mathbb{N}$,

$$x_n = \left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n = \frac{(n+1)^n}{n^n}.$$

If $n \geq 2$ then, similarly, $x_{n-1} = \dfrac{n^{n-1}}{(n-1)^{n-1}}$. Hence

$$\frac{x_n}{x_{n-1}} = \frac{(n+1)^n}{n^n} \cdot \frac{(n-1)^{n-1}}{n^{n-1}} = \left(\frac{(n+1)(n-1)}{n^2}\right)^{n-1} \cdot \frac{n+1}{n}$$

$$= \left(\frac{n^2-1}{n^2}\right)^{n-1} \cdot \frac{n+1}{n} = \left(1 - \frac{1}{n^2}\right)^{n-1} \left(1 + \frac{1}{n}\right).$$

To proceed, we need the following estimate.

Lemma If $0 < x < 1$, then $(1-x)^k \geq 1 - kx$ for all $k \in \mathbb{N}$.

Using the lemma, we obtain that

$$\frac{x_n}{x_{n-1}} = \left(1 - \frac{1}{n^2}\right)^{n-1} \left(1 + \frac{1}{n}\right) \geq \left(1 - \frac{n-1}{n^2}\right) \left(1 + \frac{1}{n}\right)$$

$$= 1 - \frac{n-1}{n^2} + \frac{1}{n} - \frac{n-1}{n^3} = 1 + \frac{1}{n^2} - \frac{n-1}{n^3} = 1 + \frac{1}{n^3} > 1.$$

Thus the sequence $\{x_n\}$ is strictly increasing.

Proof of the lemma: The lemma is proved by induction on k. The case $k = 1$ is trivial as $(1-x)^1 = 1 - 1 \cdot x$. Now assume that the inequality $(1-x)^k \geq 1 - kx$ holds for some $k \in \mathbb{N}$ and all $x \in (0,1)$. Then $(1-x)^{k+1} = (1-x)^k(1-x)$ $\geq (1-kx)(1-x) = 1 - kx - x + kx^2 > 1 - (k+1)x$. ∎

Remark. According to the Binomial Formula,

$$(1-x)^k = 1 - kx + \tfrac{k(k-1)}{2}x^2 - \dots$$

Now let us show that the sequence $\{x_n\}$ is bounded. Since $\{x_n\}$ is increasing, it is enough to show that it is bounded above. By the Binomial Formula,

$$x_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \left(\frac{1}{n}\right)^k.$$

Observe that $\dfrac{n!}{(n-k)!} \left(\dfrac{1}{n}\right)^k \leq 1$ for all k, $0 \leq k \leq n$.

It follows that $x_n \leq \sum_{k=0}^{n} \dfrac{1}{k!} = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}$.

Further observe that $k! \geq 2^{k-1}$ for all $k \geq 0$. Therefore we obtain

$$x_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 3 - \frac{1}{2^{n-1}} < 3.$$

Cauchy sequences

Definition. A sequence $\{x_n\}$ of real numbers is called a **Cauchy sequence** if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ whenever $n, m \geq N$.

Theorem Any convergent sequence is Cauchy.

Proof: Let $\{x_n\}$ be a convergent sequence and a be its limit. Then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon/2$ whenever $n \geq N$. Now for any natural numbers $n, m \geq N$ we have

$$|x_n - x_m| = |x_n - a + a - x_m| \leq |x_n - a| + |x_m - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence.

Theorem Any Cauchy sequence is convergent.

Proof: Suppose $\{x_n\}$ is a Cauchy sequence. First let us show that this sequence is bounded. Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < 1$ whenever $n, m \geq N$. In particular, $|x_n - x_N| < 1$ for all $n \geq N$. Then $|x_n| = |(x_n - x_N) + x_N| \leq |x_n - x_N| + |x_N| < |x_N| + 1$. It follows that for any $n \in \mathbb{N}$ we have $|x_n| \leq M$, where $M = \max(|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1)$.

Now the Bolzano-Weierstrass theorem implies that $\{x_n\}$ has a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converging to some $a \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $K_\varepsilon \in \mathbb{N}$ such that $|x_{n_k} - a| < \varepsilon/2$ for all $k \geq K_\varepsilon$. Also, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon/2$ whenever $n, m \geq N_\varepsilon$. Let $k = \max(K_\varepsilon, N_\varepsilon)$. Then $k \geq K_\varepsilon$ and $n_k \geq k \geq N_\varepsilon$. Therefore for any $n \geq N_\varepsilon$ we obtain $|x_n - a| = |(x_n - x_{n_k}) + (x_{n_k} - a)| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus the entire sequence $\{x_n\}$ converges to a.

Limit points

Definition. A **limit point** of a sequence $\{x_n\}$ is the limit of any convergent subsequence of $\{x_n\}$.

Examples and properties.

- A convergent sequence has only one limit point, its limit.
- Any bounded sequence has at least one limit point.
- If a bounded sequence is not convergent, then it has at least two limit points.
- The sequence $\{(-1)^n\}$ has two limit points, 1 and -1.
- If all elements of a sequence belong to a closed interval $[a, b]$, then all its limit points belong to $[a, b]$ as well.
- The set of limit points of the sequence $\{\sin n\}$ is the entire interval $[-1, 1]$.
- If a sequence diverges to infinity, then it has no limit points.
- If a sequence does not diverge to infinity, then it has a bounded subsequence and hence it has a limit point.