MATH 409

Advanced Calculus I

Lecture 10:
Continuity.
Properties of continuous functions.

Continuity

Definition. Given a set $E \subset \mathbb{R}$, a function $f : E \to \mathbb{R}$, and a point $c \in E$, the function f is **continuous at** c if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - c| < \delta$ and $x \in E$ imply $|f(x) - f(c)| < \varepsilon$.

We say that the function f is **continuous on** a set $E_0 \subset E$ if f is continuous at every point $c \in E_0$. The function f is **continuous** if it is continuous on the entire domain E.

Remarks. • In the case $E = (a, b)$, the function f is continuous at a point $c \in E$ if and only if $f(c) = \lim_{x \to c} f(x)$.

• In the case $E = [a, b]$, the function f is continuous at a point $c \in (a, b)$ if $f(c) = \lim_{x \to c} f(x)$. It is continuous at a if $f(a) = \lim_{x \to a+} f(x)$ and continuous at b if $f(b) = \lim_{x \to b-} f(x)$.

Theorem A function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$ if and only if for any sequence $\{x_n\}$ of elements of E, $x_n \to c$ as $n \to \infty$ implies $f(x_n) \to f(c)$ as $n \to \infty$.

Theorem Suppose that functions $f, g : E \to \mathbb{R}$ are both continuous at a point $c \in E$. Then the functions $f + g$, $f - g$, and fg are also continuous at c. If, additionally, $g(c) \neq 0$, then the function f/g is continuous at c as well.

Bounded functions

Definition. A function $f : E \to \mathbb{R}$ is **bounded on a subset** $E_0 \subset E$ if there exists $C > 0$ such that $|f(x)| \leq C$ for all $x \in E_0$. In the case $E_0 = E$, we say that f is **bounded**.

The function f is **bounded above** on E_0 if there exists $C \in \mathbb{R}$ such that $f(x) \leq C$ for all $x \in E_0$. It is **bounded below** on E_0 if there exists $C \in \mathbb{R}$ such that $f(x) \geq C$ for all $x \in E_0$.

Equivalently, f is bounded on E_0 if the image $f(E_0)$ is a bounded subset of $\mathbb{R}$. Likewise, the function f is bounded above on E_0 if the image $f(E_0)$ is bounded above. It is bounded below on E_0 if $f(E_0)$ is bounded below.

Example. $h : \mathbb{R} \to \mathbb{R}$, $h(0) = 0$, $h(x) = 1/x$ for $x \neq 0$.

The function h is unbounded. At the same time, it is bounded on $[1, \infty)$ and on $(-\infty, -1]$. It is bounded below on $(0, \infty)$ and bounded above on $(-\infty, 0)$.

Theorem If $I = [a, b]$ is a closed, bounded interval of the real line, then any continuous function $f : I \to \mathbb{R}$ is bounded.

Proof: Assume that a function $f : I \to \mathbb{R}$ is unbounded. Then for every $n \in \mathbb{N}$ there exists a point $x_n \in I$ such that $|f(x_n)| > n$. We obtain a sequence $\{x_n\}$ of elements of I such that the sequence $\{f(x_n)\}$ diverges to infinity.

Since the interval I is bounded, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ (due to the Bolzano-Weierstrass Theorem). Let $c = \lim_{k\to\infty} x_{n_k}$. Then $c \in [a, b]$ (due to the Comparison Theorem). Since the sequence $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$, it diverges to infinity. In particular, it does not converge to $f(c)$. It follows that the function f is discontinuous at c.

Thus any continuous function on $[a, b]$ has to be bounded.

Discontinuities

A function $f : E \to \mathbb{R}$ is **discontinuous** at a point $c \in E$ if it is not continuous at c. There are various kinds of discontinuities including the following ones.

- The function f has a **jump discontinuity** at a point c if both one-sided limits at c exist, but they are not equal: $\lim_{x \to c-} f(x) \neq \lim_{x \to c+} f(x)$.

- The function f has a **removable discontinuity** at a point c if the limit at c exists, but it is different from the value at c: $\lim_{x \to c} f(x) \neq f(c)$.

- If the function f is continuous at a point c, then it is locally bounded at c, which means that f is bounded on the set $(c - \delta, c + \delta) \cap E$ provided $\delta > 0$ is small enough. Hence any function **not locally bounded** at c is discontinuous at c.

Examples

- Constant function: $f(x) = a$ for all $x \in \mathbb{R}$ and some $a \in \mathbb{R}$.

Since $\lim_{x \to c} f(x) = a$ for all $c \in \mathbb{R}$, the function f is continuous.

- Identity function: $f(x) = x$, $x \in \mathbb{R}$.

Since $\lim_{x \to c} f(x) = c$ for all $c \in \mathbb{R}$, the function is continuous.

- Step function: $f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$

Since $\lim_{x \to 0-} f(x) = 0$ and $\lim_{x \to 0+} f(x) = 1$, the function has a jump discontinuity at 0. It is continuous on $\mathbb{R} \setminus \{0\}$.

Examples

- $f(0) = 0$ and $f(x) = \dfrac{1}{x}$ for $x \neq 0$.

Since $\lim_{x \to c} f(x) = 1/c$ for all $c \neq 0$, the function f is continuous on $\mathbb{R} \setminus \{0\}$. It is discontinuous at 0 as it is not locally bounded at 0.

- $f(0) = 0$ and $f(x) = \sin \dfrac{1}{x}$ for $x \neq 0$.

Since $\lim_{x \to 0+} f(x)$ does not exist, the function is discontinuous at 0. Notice that it is neither jump nor removable discontinuity, and the function f is bounded.

- $f(0) = 0$ and $f(x) = x \sin \dfrac{1}{x}$ for $x \neq 0$.

Since $\lim_{x \to 0} f(x) = 0$, the function is continuous at 0.

Examples

- Dirichlet function: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Since $\lim_{x \to c} f(x)$ never exists, the function has no points of continuity.

- Riemann function:

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q, \text{ a reduced fraction,} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Since $\lim_{x \to c} f(x) = 0$ for all $c \in \mathbb{R}$, the function f is continuous at irrational points and discontinuous at rational points. Moreover, all discontinuities are removable.

Extreme Value Theorem

Theorem If $I = [a, b]$ is a closed, bounded interval of the real line, then any continuous function $f : I \to \mathbb{R}$ attains its extreme values (maximum and minimum) on I. To be precise, there exist points $x_{\min}, x_{\max} \in I$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \text{ for all } x \in I.$$

Remark 1. The theorem may not hold if the interval I is not closed. Counterexample: $f(x) = x$, $x \in (0, 1)$. Neither maximum nor minimum is attained.

Remark 2. The theorem may not hold if the interval I is not bounded. Counterexample: $f(x) = 1/(1 + x^2)$, $x \in [0, \infty)$. The maximal value is attained at 0 but the minimal value is not attained.

Extreme Value Theorem

Proof of the theorem: Since the function f is continuous, it is bounded on I. Hence $m = \inf_{x \in I} f(x)$ and $M = \sup_{x \in I} f(x)$ are well-defined numbers. In different notation: $m = \inf f(I)$, $M = \sup f(I)$. Clearly, $m \leq f(x) \leq M$ for all $x \in I$.

For any $n \in \mathbb{N}$ the number $M - \frac{1}{n}$ is not an upper bound of the set $f(I)$ while $m + \frac{1}{n}$ is not a lower bound of $f(I)$. Hence we can find points $x_n, y_n \in I$ such that $f(x_n) > M - \frac{1}{n}$ and $f(y_n) < m + \frac{1}{n}$. By construction, $f(x_n) \to M$ and $f(y_n) \to m$ as $n \to \infty$. The Bolzano-Weierstrass Theorem implies that the sequences $\{x_n\}$ and $\{y_n\}$ have convergent subsequences (or, in other words, they have limit points). Let c be a limit point of $\{x_n\}$ and d be a limit point of $\{y_n\}$. Notice that $c, d \in I$. The continuity of f implies that $f(c)$ is a limit point of $\{f(x_n)\}$ and $f(d)$ is a limit point of $\{f(y_n)\}$. We conclude that $f(c) = M$ and $f(d) = m$.

Intermediate Value Theorem

Theorem If a function $f : [a, b] \to \mathbb{R}$ is continuous then any number y_0 that lies between $f(a)$ and $f(b)$ is a value of f, i.e., $y_0 = f(x_0)$ for some $x_0 \in [a, b]$.

Proof: In the case $f(a) = f(b)$, the theorem is trivial. In the case $f(a) > f(b)$, we notice that the function $-f$ is continuous on $[a, b]$, $-f(a) < -f(b)$, and $-y_0$ lies between $-f(a)$ and $-f(b)$. Hence we can assume without loss of generality that $f(a) < f(b)$.

Further, if a number y_0 lies between $f(a)$ and $f(b)$, then 0 lies between $f(a) - y_0$ and $f(b) - y_0$. Moreover, the function $g(x) = f(x) - y_0$ is continuous on $[a, b]$ and $g(a) < g(b)$ if and only if $f(a) < f(b)$. Hence it is no loss to assume that $y_0 = 0$.

Now the theorem is reduced to the following special case.

Theorem If a function $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then $f(x_0) = 0$ for some $x_0 \in (a, b)$.

Proof: Let $E = \{x \in [a, b] \mid f(x) > 0\}$. The set E is nonempty (as $b \in E$) and bounded (as $E \subset [a, b]$). Therefore $x_0 = \inf E$ exists. Observe that $x_0 \in [a, b]$ ($x_0 \leq b$ as $b \in E$; $x_0 \geq a$ as a is a lower bound of E). To complete the proof, we need the following lemma.

Lemma If a function f is continuous at a point c and $f(c) \neq 0$, then f maintains its sign in a sufficiently small neighborhood of c.

The lemma implies that $f(x_0) = 0$. Indeed, if $f(x_0) \neq 0$ then for some $\delta > 0$ the function f maintains its sign in the interval $(x_0 - \delta, x_0 + \delta) \cap [a, b]$. In the case $f(x_0) > 0$, we obtain that $x_0 > a$ and x_0 is not a lower bound of E. In the case $f(x_0) < 0$, we obtain that $x_0 < b$ and x_0 is not the greatest lower bound of E. Either way we arrive at a contradiction.

Lemma If a function f is continuous at a point c and $f(c) \neq 0$, then f maintains its sign in a sufficiently small neighborhood of c.

Proof of lemma: Since f is continuous at c and $|f(c)| > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < |f(c)|$ whenever $|x - c| < \delta$. The inequality $|f(x) - f(c)| < |f(c)|$ implies that the number $f(x)$ has the same sign as $f(c)$. ■

Corollary If a real-valued function f is continuous on a closed bounded interval $[a, b]$, then the image $f([a, b])$ is also a closed bounded interval.

Proof: By the Extreme Value Theorem, there exist points $x_{\min}, x_{\max} \in [a, b]$ such that $f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for all $x \in [a, b]$. Let I_0 denote the closed interval with endpoints $x_{\min}$ and $x_{\max}$. Let J denote the closed interval with endpoints $f(x_{\min})$ and $f(x_{\max})$. We have that $f([a, b]) \subset J$. The Intermediate Value Theorem implies that $f(I_0) = J$. Since $I_0 \subset [a, b]$, we obtain that $f([a, b]) = J$.