MATH 409

Advanced Calculus I

Lecture 14:
The derivative.
Differentiability theorems.

The derivative

Definition. A real function f is said to be **differentiable** at a point $a \in \mathbb{R}$ if it is defined on an open interval containing a and the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The limit is denoted $f'(a)$ and called the **derivative** of f at a.

An equivalent condition is $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

Remark. The one-sided limits $\lim_{x \to a+} \frac{f(x)-f(a)}{x-a}$ and $\lim_{x \to a-} \frac{f(x)-f(a)}{x-a}$ are called the right-hand and left-hand derivatives of f at a. One of them or both might exist even if f is not differentiable at a.

Examples

- Constant function: $f(x) = c, \ \ x \in \mathbb{R}$.

$$\frac{f(x+h) - f(x)}{h} = \frac{c - c}{h} = 0 \ \text{ for all } \ x \in \mathbb{R} \ \text{ and } \ h \neq 0.$$

Therefore $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0$.

That is, f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$.

- Identity function: $f(x) = x, \ \ x \in \mathbb{R}$.

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h) - x}{h} = 1 \ \text{ for all } \ x \in \mathbb{R}, \ h \neq 0.$$

Therefore $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 1$.

That is, f is differentiable on $\mathbb{R}$ and $f'(x) = 1$ for all $x \in \mathbb{R}$.

Examples

- $f(x) = x^2, \; x \in \mathbb{R}$.

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h.$$

Therefore $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2x + h) = 2x.$

That is, f is differentiable on $\mathbb{R}$ and $f'(x) = 2x$ for all $x \in \mathbb{R}$.

Examples

- $f(x) = \dfrac{1}{x}, \quad x \in (-\infty, 0) \cup (0, \infty).$

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h} \cdot \left(\frac{1}{x+h} - \frac{1}{x} \right)$$
$$= \frac{1}{h} \cdot \frac{x - (x+h)}{(x+h)x} = -\frac{1}{(x+h)x}.$$

Therefore $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} -\frac{1}{(x+h)x} = -\frac{1}{x^2}.$

That is, f is differentiable on $\mathbb{R} \setminus \{0\}$ and $f'(x) = -1/x^2$ for all $x \neq 0$.

Examples

- $f(x) = \sqrt{x}, \;\; x \in [0, \infty)$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

In the case $x > 0$,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

In the case $x = 0$, $\displaystyle \lim_{h \to 0+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0+} \frac{1}{\sqrt{h}} = +\infty.$

Hence f is differentiable on $(0, \infty)$ and $f'(x) = 1/(2\sqrt{x})$ for all $x > 0$. It is not differentiable at 0.

Examples

- $f(x) = \sin x, \quad x \in \mathbb{R}.$

Using the formula $\sin\alpha - \sin\beta = 2\,\sin\dfrac{\alpha-\beta}{2}\,\cos\dfrac{\alpha+\beta}{2}$,
we obtain

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin x}{h} = \frac{2}{h}\,\sin\frac{h}{2}\,\cos\frac{2x+h}{2}.$$

Therefore $\displaystyle \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{2}{h}\,\sin\frac{h}{2}\,\cos\frac{2x+h}{2}$

$$= \lim_{h\to 0}\frac{\sin(h/2)}{h/2}\cdot\lim_{h\to 0}\cos(x+h/2) = 1\cdot\cos x = \cos x.$$

That is, f is differentiable on $\mathbb{R}$ and $f'(x) = \cos x$ for all $x \in \mathbb{R}$.

Differentiability $\Longrightarrow$ continuity

Theorem If a function f is differentiable at a point a, then it is continuous at a.

Proof:

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(f(a) + \frac{f(x)-f(a)}{x-a}(x-a)\right)$$

$$= \lim_{x\to a} f(a) + \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot \lim_{x\to a}(x-a)$$

$$= f(a) + f'(a)\cdot 0 = f(a).$$

Remark. Similarly, if f has a right-hand derivative at a, then $\lim\limits_{x\to a+} f(x) = f(a)$. If f has a left-hand derivative at a, then $\lim\limits_{x\to a-} f(x) = f(a)$.

Sum Rule and Homogeneous Rule

Theorem If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the sum $f + g$ is also differentiable at a. Moreover, $(f+g)'(a) = f'(a) + g'(a)$.

Proof: $$\lim_{x \to a} \frac{(f+g)(x) - (f+g)(a)}{x-a}$$
$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} + \lim_{x \to a} \frac{g(x) - g(a)}{x-a} = f'(a) + g'(a).$$

Theorem If a function f is differentiable at a point $a \in \mathbb{R}$, then for any $r \in \mathbb{R}$ the scalar multiple rf is also differentiable at a. Moreover, $(rf)'(a) = rf'(a)$.

Proof: $$\lim_{x \to a} \frac{(rf)(x) - (rf)(a)}{x-a} = \lim_{x \to a} r\frac{f(x) - f(a)}{x-a} = rf'(a).$$

Product Rule

Theorem If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the product $f \cdot g$ is also differentiable at a. Moreover, $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$.

Proof: Since f and g are differentiable at the point a, there is an open interval $I = (c, d)$ containing a such that both f and g are defined on I. For every $x \in I \setminus \{a\}$ we have

$$f(x)g(x) - f(a)g(a) = f(x)g(x) - f(a)g(x) + f(a)g(x) \\ - f(a)g(a) = \big(f(x) - f(a)\big)g(x) + f(a)\big(g(x) - g(a)\big).$$

Then $\frac{(f\cdot g)(x)-(f\cdot g)(a)}{x-a} = \frac{f(x)-f(a)}{x-a}g(x) + f(a)\frac{g(x)-g(a)}{x-a}$ so that

$$\lim_{x\to a} \frac{(f \cdot g)(x) - (f \cdot g)(a)}{x - a} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a} g(x) \\ + \lim_{x\to a} f(a) \cdot \lim_{x\to a} \frac{g(x) - g(a)}{x - a} = f'(a)g(a) + f(a)g'(a).$$

Reciprocal Rule

Theorem If a function f is differentiable at a point $a \in \mathbb{R}$ and $f(a) \neq 0$, then the function $1/f$ is also differentiable at a. Moreover, $(1/f)'(a) = -f'(a)/f^2(a)$.

Proof: The function f is defined on an open interval (c, d) containing a. We know that f is continuous at a. Since $\varepsilon = |f(a)| > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for any $x \in I = (c, d) \cap (a - \delta, a + \delta)$. Then $f(x) \neq 0$ for all $x \in I$. In particular, the function $1/f$ is defined on I, an open interval containing a. Now

$$\lim_{x \to a} \frac{(1/f)(x) - (1/f)(a)}{x - a} = \lim_{x \to a} \left(\frac{1}{f(x)} - \frac{1}{f(a)} \right) \frac{1}{x - a}$$

$$= \lim_{x \to a} \frac{f(a) - f(x)}{f(x)f(a)} \cdot \frac{1}{x - a} = \lim_{x \to a} \left(-\frac{f(x) - f(a)}{x - a} \cdot \frac{1}{f(x)f(a)} \right)$$

$$= -\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} \frac{1}{f(x)f(a)} = -\frac{f'(a)}{f^2(a)}.$$

Difference Rule and Quotient Rule

Theorem If functions f and g are differentiable at a point $a \in \mathbb{R}$, then the difference $f - g$ is also differentiable at a. Moreover, $(f - g)'(a) = f'(a) - g'(a)$.

Proof: By the Homogeneous Rule, the function $-g = (-1)g$ is differentiable at a and $(-g)'(a) = -g'(a)$. By the Sum Rule, the function $f - g = f + (-g)$ is also differentiable at a and $(f - g)'(a) = f'(a) + (-g)'(a) = f'(a) - g'(a)$.

Theorem If functions f and g are differentiable at $a \in \mathbb{R}$ and $g(a) \neq 0$, then the quotient f/g is also differentiable at a. Moreover, $(\frac{f}{g})'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$.

Proof: By the Reciprocal Rule, the function $1/g$ is differentiable at a and $(1/g)'(a) = -g'(a)/g^2(a)$. By the Product Rule, the function $f/g = f \cdot (1/g)$ is also differentiable at a and $(f/g)'(a) = f'(a)/g(a) + f(a)(1/g)'(a)$ $= \big(f'(a)g(a) - f(a)g'(a)\big)/g^2(a)$.

Chain Rule

Theorem If a function f is differentiable at a point $a \in \mathbb{R}$ and a function g is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a. Moreover, $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Proof: The function f is defined on an open interval $I = (a - \delta, a + \delta)$ while g is defined on an open interval $J = (f(a) - \varepsilon, f(a) + \varepsilon)$. Since f is continuous at a, there exists $\delta_0 \in (0, \delta)$ such that $f(I_0) \subset J$, where $I_0 = (a - \delta_0, a + \delta_0)$. Then $g \circ f$ is defined on I_0. For any $x \in I_0$ such that $f(x) \neq f(a)$,

$$\frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a}.$$

This implies the Chain Rule unless there is a sequence $\{x_n\}$ converging to a such that $x_n \neq a$ while $f(x_n) = f(a)$. In this case, one can show that $(g \circ f)'(a) = f'(a) = 0$.

Examples

- $f(x) = \cos x, \quad x \in \mathbb{R}.$

The function f can be represented as a composition $f = h \circ g$, where $g(x) = x + \pi/2$ and $h(x) = \sin x$, $x \in \mathbb{R}$. Since $g'(x) = 1$ and $h'(x) = \cos x$ for all $x \in \mathbb{R}$, the Chain Rule implies that f is differentiable on $\mathbb{R}$ and $f'(x) = h'(g(x))g'(x) = \cos(x + \pi/2) = -\sin x$ for all $x \in \mathbb{R}$.

- $f(x) = \tan x, \quad x \in (-\pi/2, \pi/2).$

Since $f(x) = \sin x/\cos x$ and $\cos x \neq 0$ for all $x \in (-\pi/2, \pi/2)$, the Quotient Rule implies that f is differentiable on $(-\pi/2, \pi/2)$ and

$$f'(x) = \frac{(\sin x)' \cos x - \sin x\,(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

for all $x \in (\pi/2, \pi/2)$.