MATH 409

Advanced Calculus I

Lecture 23: Convergence of infinite series.

Infinite series

Definition. Given a sequence $\{a_n\}$ of real numbers, an expression $a_1 + a_2 + \cdots + a_n + \dots$ or $\sum_{n=1}^{\infty} a_n$ is called an **infinite series** with **terms** a_n. The **partial sum** of order n of the series is defined by $s_n = a_1 + a_2 + \cdots + a_n$. If the sequence $\{s_n\}$ converges to a limit $s \in \mathbb{R}$, we say that the series **converges** to s or that s is the **sum** of the series and write $\sum_{n=1}^{\infty} a_n = s$. Otherwise the series **diverges**.

Theorem (Cauchy Criterion) An infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $|a_n + a_{n+1} + \cdots + a_m| < \varepsilon$.

Proof: Let $\{s_n\}$ be the sequence of partial sums. Then $a_n + a_{n+1} + \cdots + a_m = s_m - s_{n-1}$. Consequently, the condition of the theorem is equivalent to the condition that $\{s_n\}$ be a Cauchy sequence. As we know, a sequence is convergent if and only if it is a Cauchy sequence.

Examples

- $$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots = 1.$$

The partial sums s_n of this series satisfy $s_n = 1 - 2^{-n}$ for all $n \in \mathbb{N}$. Thus $s_n \to 1$ as $n \to \infty$.

- $$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots = 1.$$

Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, the partial sums s_n of this series satisfy $s_n = 1 - \frac{1}{n+1}$. Thus $s_n \to 1$ as $n \to \infty$.

- $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + \dots$ diverges.

The partial sums s_n satisfy $s_n = -1$ for odd n and $s_n = 0$ for even n. Hence the sequence $\{s_n\}$ has no limit.

Some properties of infinite series

Theorem (Divergence Test) If the terms of an infinite series do not converge to zero, then the series diverges.

Theorem If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

and

$$\sum_{n=1}^{\infty}(ra_n) = r\sum_{n=1}^{\infty} a_n$$

for any $r \in \mathbb{R}$.

Theorem If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

Example

- The **geometric series** $\sum_{n=0}^{\infty} x^n$ converges if and only if $|x| < 1$, in which case its sum is $\dfrac{1}{1-x}$.

In the case $|x| \geq 1$, the series fails the Divergence Test. For any $x \neq 1$, the partial sums s_n of the geometric series satisfy

$$s_n = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

In the case $|x| < 1$, we obtain that $s_n \to 1/(1-x)$ as $n \to \infty$.

Series with nonnegative terms

Suppose that a series $\sum_{n=1}^{\infty} a_n$ has nonnegative terms, $a_n \geq 0$ for all $n \in \mathbb{N}$. Then the sequence of partial sums $s_n = a_1 + a_2 + \cdots + a_n$ is increasing. It follows that $\{s_n\}$ converges to a finite limit if bounded and diverges to $+\infty$ otherwise. In the latter case, we write $\sum_{n=1}^{\infty} a_n = \infty$.

Theorem (Comparison Test) Suppose that $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n \leq b_n$ for large n. Then convergence of the series $\sum_{n=1}^{\infty} b_n$ implies convergence of $\sum_{n=1}^{\infty} a_n$ while $\sum_{n=1}^{\infty} a_n = \infty$ implies $\sum_{n=1}^{\infty} b_n = \infty$.

Proof: Since changing a finite number of terms does not affect convergence of a series, it is no loss to assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then the partial sums $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ satisfy $s_n \leq t_n$ for all n. Consequently, if $s_n \to +\infty$ as $n \to \infty$, then also $t_n \to +\infty$ as $n \to \infty$. Conversely, if $\{t_n\}$ is bounded, then so is $\{s_n\}$.

Integral test

Theorem Suppose that a function $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Then

(i) a sequence $\{y_n\}$ is bounded, where

$$y_n = f(1) + f(2) + \cdots + f(n) - \int_1^n f(x)\,dx, \quad n = 1, 2, \dots$$

(ii) the series $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if the function f is improperly integrable on $[1, \infty)$.

To prove the theorem, we need the following lemma.

Lemma Any monotone function $g : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$.

Idea of the proof: Any monotone function has only jump discontinuities. Further, any function has at most countably many jump discontinuities. Besides, a monotone function on $[a, b]$ is clearly bounded.

Proof of the theorem: The lemma implies that the function f is integrable on every closed interval $J = [a, b] \subset [1, \infty)$. Then for any partition P of the interval J the lower Darboux sum $L(f, P)$ and the upper Darboux sum $U(f, P)$ satisfy

$$L(f, P) \leq \int_a^b f(x)\, dx \leq U(f, P).$$

Let $P = \{x_0, x_1, \ldots, x_k\}$, where $x_0 < x_1 < \cdots < x_k$. Then $\sup f([x_{j-1}, x_j]) = f(x_{j-1})$ and $\inf f([x_{j-1}, x_j]) = f(x_j)$ since f is decreasing. In the case $J = [1, n]$, where $n \in \mathbb{N}$, and $P = \{1, 2, \ldots, n\}$ we obtain $L(f, P) = f(2) + f(3) + \ldots + f(n)$, $U(f, P) = f(1) + f(2) + \cdots + f(n-1)$. Then the above inequalities imply that $0 < f(n) \leq y_n \leq f(1)$. Thus the sequence $\{y_n\}$ is bounded.

Since f is positive, the series $\sum_{n=1}^{\infty} f(n)$ either converges or else it diverges to $+\infty$. Likewise the improper integral $\int_1^{\infty} f(x)\, dx$ either converges or else it diverges to $+\infty$. Since the sequence $\{y_n\}$ is bounded, divergence of the series and the integral imply each other.

Examples

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for any $p > 1$ and divergent for any $p < 1$.

For any $p \neq 1$ we have $\int x^{-p}\, dx = x^{1-p}/(1-p) + C$ on the interval $[1, \infty)$. The antiderivative converges to a finite limit at $+\infty$ in the case $p > 1$ and diverges to $+\infty$ in the case $p < 1$. Hence the function $f(x) = x^{-p}$ is improperly integrable on $[1, \infty)$ for $p > 1$ but not for $p < 1$. By the Integral Test, the series is convergent for $p > 1$ and divergent for $0 \leq p < 1$. If $p < 0$ then the Integral Test does not apply since f is not decreasing. In this case, the series is divergent since the terms $1/n^p$ do not converge to 0 as $n \to \infty$.

Examples

- The **harmonic series** $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

Indeed, $\int_1^n x^{-1}\,dx = \log n \to +\infty$ as $n \to \infty$. By the Integral Test, the series is divergent. Moreover, the sequence $y_n = \sum_{k=1}^{n} k^{-1} - \log n$ is bounded (actually, it is decreasing and hence convergent).

- $\sum_{n=2}^{\infty} \dfrac{1}{n \log^2 n}$ converges.

The antiderivative of $f(x) = (x \log^2 x)^{-1}$ on $(1, \infty)$ is

$$\int \frac{dx}{x \log^2 x} = \int \frac{d(\log x)}{\log^2 x} = -\frac{1}{\log x} + C.$$

Since the antiderivative converges to a finite limit at $+\infty$, the function f is improperly integrable on $[2, \infty)$. By the Integral Test, the series converges.

Examples

- $\sum_{n=1}^{\infty} \dfrac{1}{1+n^2}$ converges.

Indeed, $0 < 1/(1+n^2) < 1/n^2$ for all $n \in \mathbb{N}$. Since the series $\sum_{n=1}^{\infty} 1/n^2$ is convergent, it remains to apply the Comparison Test. Alternatively, we can use the Integral Test. Indeed, $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$ converges to a finite limit at $+\infty$ so that the function $f(x) = 1/(1+x^2)$ is improperly integrable on $[1, \infty)$.

- $\sum_{n=1}^{\infty} e^{-n^2}$ converges.

We have $0 < e^{-n^2} \leq e^{-n}$ for all $n \in \mathbb{N}$. The geometric series $\sum_{n=1}^{\infty} e^{-n}$ is convergent since $0 < e^{-1} < 1$. By the Comparison Test, $\sum_{n=1}^{\infty} e^{-n^2}$ is convergent as well.