MATH 409

Advanced Calculus I

Lecture 24:
Alternating series.
Absolute convergence of series.

Some tests of convergence

[Divergence Test] If the terms of an infinite series do not converge to zero, then the series diverges.

[Cauchy Criterion] An infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $|a_n + a_{n+1} + \cdots + a_m| < \varepsilon$.

[Comparison Test] Suppose that $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n \leq b_n$ for large n. Then convergence of the series $\sum_{n=1}^{\infty} b_n$ implies convergence of $\sum_{n=1}^{\infty} a_n$ while $\sum_{n=1}^{\infty} a_n = \infty$ implies $\sum_{n=1}^{\infty} b_n = \infty$.

[Integral Test] Suppose that a function $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the function f is improperly integrable on $[1, \infty)$.

Alternating Series Test

Definition. An infinite series $\sum_{n=1}^{\infty} a_n$ is called **alternating** if any two neighboring terms have different signs: $a_n a_{n+1} < 0$ for all $n \in \mathbb{N}$.

Theorem (Leibniz Criterion) If $\{a_n\}$ is a decreasing sequence of positive numbers and $a_n \to 0$ as $n \to \infty$, then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

converges.

Theorem (Leibniz Criterion) If $\{a_n\}$ is a decreasing sequence of positive numbers and $a_n \to 0$ as $n \to \infty$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$ converges.

Proof: Let $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ be the partial sum of order n of the series. For any $n \in \mathbb{N}$ we have

$$s_{2n} = s_{2n-1} - a_{2n} < s_{2n-1}.$$

Since the sequence $\{a_n\}$ is decreasing, we also have

$$s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} \leq s_{2n-1},$$
$$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2} \geq s_{2n}.$$

Therefore $s_{2n} \leq s_{2n+2} < s_{2n+1} \leq s_{2n-1}$ for all $n \in \mathbb{N}$. It follows that a subsequence $\{s_{2n}\}$ is increasing, a subsequence $\{s_{2n-1}\}$ is decreasing, and both are bounded. Hence both subsequences are convergent. Since $s_{2n-1} - s_{2n} = a_{2n} \to 0$ as $n \to \infty$, both subsequences converge to the same limit L. Then L is the limit of the entire sequence $\{s_n\}$.

Examples

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

  The series converges due to the Alternating Series Test. One can show that the sum is $\log 2$.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2n-1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots$

  After multiplying all terms by -1, the series satisfy all conditions of the Alternating Series Test. It follows that the series converges. One can show that the sum is $-\pi/4$.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n}{2n-1} = 1 - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \dots$

  The series is alternating and the terms decrease in absolute value. However the absolute values of terms converge to $1/2$ instead of 0. Hence the series diverges.

Absolute convergence

Definition. An infinite series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if $\sum_{n=1}^{\infty} |a_n| < \infty$.

Theorem Any absolutely convergent series is convergent.

Proof: Suppose that a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, i.e., the series $\sum_{n=1}^{\infty} |a_n|$ converges. By the Cauchy Criterion, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\big| |a_n| + |a_{n+1}| + \cdots + |a_m| \big| < \varepsilon$$

for $m \geq n \geq N$. Then

$$|a_n + a_{n+1} + \cdots + a_m| \leq |a_n| + |a_{n+1}| + \cdots + |a_m| < \varepsilon$$

for $m \geq n \geq N$. According to the Cauchy Criterion, the series $\sum_{n=1}^{\infty} a_n$ converges.

Examples

- $\sum_{n=1}^{\infty} \dfrac{1}{n^3} = 1 + \dfrac{1}{2^3} + \dfrac{1}{3^3} + \dfrac{1}{4^3} + \dots$

  The series converges due to the Integral Test. Since it has positive terms, it is absolutely convergent as well.

- $\sum_{n=1}^{\infty} \dfrac{\sin n}{n^2} = \sin 1 + \dfrac{\sin 2}{4} + \dfrac{\sin 3}{9} + \dfrac{\sin 4}{16} + \dots$

  Since $|\sin(n)/n^2| \le 1/n^2$ and the series $\sum_{n=1}^{\infty} 1/n^2$ converges, this series converges absolutely due to the Comparison Test.

- $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dots$

  The series converges (due to the Alternating Series Test), but not absolutely as the series $\sum_{n=1}^{\infty} 1/n$ diverges.

Ratio Test a.k.a. d'Alembert's Criterion

Theorem Let $\{a_n\}$ be a sequence of real numbers with $a_n \neq 0$ for large n. Suppose that a limit

$$r = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

exists (finite or infinite).

(i) If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
(ii) If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. In the case $r = 1$, the Ratio Test is inconclusive. For example, consider a series $\sum_{n=1}^{\infty} n^{-p}$, where $p > 0$. Then

$$r = \lim_{n\to\infty} \frac{(n+1)^{-p}}{n^{-p}} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^{p} = 1$$

for all $p > 0$. However the series converges for $p > 1$ and diverges for $0 < p \leq 1$.

Theorem Let $\{a_n\}$ be a sequence of real numbers with $a_n \neq 0$ for large n. Suppose that a limit $r = \lim_{n\to\infty} |a_{n+1}|/|a_n|$ exists (finite or infinite).

(i) If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: If $r > 1$, then $|a_{n+1}|/|a_n| > 1$ for n large enough. It follows that the sequence $\{|a_n|\}$ is eventually increasing. Then $a_n \not\to 0$ as $n \to \infty$ so that the series $\sum_{n=1}^{\infty} a_n$ diverges due to the Divergence Test.

In the case $r < 1$, choose some $x \in (r, 1)$. Then $|a_{n+1}|/|a_n| < x$ for n large enough. Consequently, $|a_{n+1}|/x^{n+1} < |a_n|/x^n$ for n large enough. That is, the sequence $\{|a_n|/x^n\}$ is eventually decreasing. It follows that this sequence is bounded. Hence $|a_n| \leq Cx^n$ for some $C > 0$ and all $n \in \mathbb{N}$. Since $0 < r < x < 1$, the geometric series $\sum_{n=1}^{\infty} x^n$ converges. So does the series $\sum_{n=1}^{\infty} Cx^n$. By the Comparison Test, the series $\sum_{n=1}^{\infty} |a_n|$ converges as well.

Root Test

Theorem Let $\{a_n\}$ be a sequence of real numbers and

$$r = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

(i) If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
(ii) If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: If $r > 1$, then $\sup_{k\geq n} \sqrt[k]{|a_k|} \geq r > 1$ for all $n \in \mathbb{N}$. Therefore for any $n \in \mathbb{N}$ there exists $k(n) \geq n$ such that $|a_{k(n)}|^{1/k(n)} > 1$. In particular, $|a_{k(n)}| > 1$. It follows that $a_k \not\to 0$ as $k \to \infty$ so that the series $\sum_{k=1}^{\infty} a_k$ diverges due to the Divergence Test.

In the case $r < 1$, choose some $x \in (r, 1)$. Then $\sup_{k\geq n} \sqrt[k]{|a_k|} < x$ for some $n \in \mathbb{N}$. This implies that $|a_k| < x^k$ for all $k \geq n$. Since $0 < r < x < 1$, the geometric series $\sum_{k=1}^{\infty} x^k$ converges. By the Comparison Test, the series $\sum_{k=1}^{\infty} |a_k|$ converges as well.

Examples

- $\sum_{n=1}^{\infty} \dfrac{n}{2^n} = \dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \dfrac{4}{16} + \dfrac{5}{32} + \dots$

If $a_n = n/2^n$, then

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \left(\frac{n}{2^n}\right)^{-1} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} \to \frac{1}{2}$$

as $n \to \infty$. By the Ratio Test, the series converges.

- $\sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots, \quad x \in \mathbb{R}.$

In the case $x = 0$, we have a finite sum. In the case $x \neq 0$, let $a_n = x^n/n!$, $n \in \mathbb{N}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \left(\frac{|x|^n}{n!}\right)^{-1} = \frac{|x|}{n+1} \to 0 \text{ as } n \to \infty.$$

By the Ratio Test, the series converges absolutely for all $x \neq 0$.

Examples

- $\sum_{n=1}^{\infty} \dfrac{(n!)^2}{(2n)!} = \dfrac{(1!)^2}{2!} + \dfrac{(2!)^2}{4!} + \dfrac{(3!)^2}{6!} + \dots$

If $a_n = (n!)^2/(2n)!$, then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n+1}{2(2n+1)} = \frac{1+n^{-1}}{4+2n^{-1}} \to \frac{1}{4}$$

as $n \to \infty$. By the Ratio Test, the series converges.

- $\sum_{n=1}^{\infty} \left(\dfrac{n}{n+1}\right)^{n^2} = \dfrac{1}{2} + \left(\dfrac{2}{3}\right)^4 + \left(\dfrac{3}{4}\right)^9 + \dots$

If $a_n = (n/(n+1))^{n^2}$, then

$$\sqrt[n]{a_n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left(1 + \frac{1}{n}\right)^{-n} \to \frac{1}{e}$$

as $n \to \infty$. By the Root Test, the series converges.