MATH 409

Advanced Calculus I

Lecture 25:
Review for the final exam.

Topics for the final

Part I: Axiomatic model of the real numbers

- Axioms of an ordered field
- Completeness axiom
- Archimedean principle
- Principle of mathematical induction
- Binomial formula
- Countable and uncountable sets

Wade's book: 1.1–1.6, Appendix A

Topics for the final

Part II: Limits and continuity

- Limits of sequences
- Limit theorems for sequences
- Monotone sequences
- Bolzano-Weierstrass theorem
- Cauchy sequences
- Limits of functions
- Limit theorems for functions
- Continuity of functions
- Extreme value and intermediate value theorems
- Uniform continuity

Wade's book: 2.1–2.5, 3.1–3.4

Topics for the final

Part III-a: Differential calculus

- Derivative of a function
- Differentiability theorems
- Derivative of the inverse function
- The mean value theorem
- Taylor's formula
- l'Hôpital's rule

Wade's book: 4.1–4.5

Topics for the final

Part III-b: Integral calculus

- Darboux sums, Riemann sums, the Riemann integral
- Properties of integrals
- The fundamental theorem of calculus
- Integration by parts
- Change of the variable in an integral
- Improper integrals, absolute integrability

Wade's book: 5.1–5.4

Topics for the final

Part IV: Infinite series

- Convergence of series
- Comparison test and integral test
- Alternating series test
- Absolute convergence
- Ratio test and root test

Wade's book: 6.1–6.4

Theorems to know

Archimedean Principle For any real number $\varepsilon > 0$ there exists a natural number n such that $n\varepsilon > 1$.

Theorem The sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{N} \times \mathbb{N}$ are countable.

Theorem The set $\mathbb{R}$ is uncountable.

Theorems on limits

Squeeze Theorem If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$ and $x_n \leq w_n \leq y_n$ for all sufficiently large n, then $\lim_{n\to\infty} w_n = a$.

Theorem Any monotone sequence converges to a limit if bounded, and diverges to infinity otherwise.

Theorem Any Cauchy sequence is convergent.

Theorems on derivatives

Theorem If functions f and g are differentiable at a point $a \in \mathbb{R}$, then their sum $f+g$, difference $f-g$, and product $f \cdot g$ are also differentiable at a. Moreover,

$$(f+g)'(a) = f'(a) + g'(a),$$
$$(f-g)'(a) = f'(a) - g'(a),$$
$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).$$

If, additionally, $g(a) \neq 0$ then the quotient f/g is also differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Mean Value Theorem If a function f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)\,(b-a)$.

Theorems on integrals

Theorem If functions f, g are integrable on an interval $[a, b]$, then the sum $f + g$ is also integrable on $[a, b]$ and

$$\int_a^b \big(f(x) + g(x)\big)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Theorem If a function f is integrable on $[a, b]$ then for any $c \in (a, b)$,

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Theorems on series

Integral Test Suppose that $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the function f is improperly integrable on $[1, \infty)$.

Ratio Test Let $\{a_n\}$ be a sequence of reals with $a_n \neq 0$ for large n. Suppose that a limit

$$r = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

exists (finite or infinite).

(i) If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Sample problems for the final exam

Problem 1 (20 pts.) Suppose $E_1, E_2, E_3, \dots$ are countable sets. Prove that their union $E_1 \cup E_2 \cup E_3 \cup \dots$ is also a countable set.

Problem 2 (20 pts.) Find the following limits:

(i) $\displaystyle\lim_{x\to 0} \log \frac{1}{1+\cot(x^2)}$, **(ii)** $\displaystyle\lim_{x\to 64} \frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$,

(iii) $\displaystyle\lim_{n\to\infty} \left(1+\frac{c}{n}\right)^n$, where $c \in \mathbb{R}$.

Sample problems for the final exam

Problem 3 (20 pts.) Prove that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

converges to $\sin x$ for any $x \in \mathbb{R}$.

Problem 4 (20 pts.) Find an indefinite integral and evaluate definite integrals:

(i) $\displaystyle\int \frac{\sqrt{1+\sqrt[4]{x}}}{2\sqrt{x}}\,dx$, **(ii)** $\displaystyle\int_0^{\sqrt{3}} \frac{x^2+6}{x^2+9}\,dx$,

(iii) $\displaystyle\int_0^{\infty} x^2 e^{-x}\,dx$.

Sample problems for the final exam

Problem 5 (20 pts.) For each of the following series, determine whether the series converges and whether it converges absolutely:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$, **(ii)** $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+2^n \cos n}{n!}$,

(iii) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$.

Sample problems for the final exam

Bonus Problem 6 (15 pts.) Prove that an infinite product

$$\prod_{n=1}^{\infty} \frac{n^2+1}{n^2} = \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{10}{9} \cdot \frac{17}{16} \cdot \ldots$$

converges, that is, partial products $\prod_{k=1}^{n} \frac{k^2+1}{k^2}$ converge to a finite limit as $n \to \infty$.

Problem 1. Suppose $E_1, E_2, E_3, \dots$ are countable sets. Prove that their union $E_1 \cup E_2 \cup \dots$ is also a countable set.

First we are going to show that the set $\mathbb{N} \times \mathbb{N}$ is countable. Consider a relation $\prec$ on the set $\mathbb{N} \times \mathbb{N}$ such that $(n_1, n_2) \prec (m_1, m_2)$ if and only if either $n_1 + n_2 < m_1 + m_2$ or else $n_1 + n_2 = m_1 + m_2$ and $n_1 < m_1$. It is easy to see that $\prec$ is a strict linear order. Moreover, for any pair $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ there are only finitely many pairs (n_1, n_2) such that $(n_1, n_2) \prec (m_1, m_2)$. It follows that $\prec$ is a well-ordering. Now we define inductively a mapping $F : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that for any $n \in \mathbb{N}$ the pair $F(n)$ is the least (relative to $\prec$) pair different from $F(k)$ for all natural numbers $k < n$. It follows from the construction that F is bijective. The inverse mapping F^{-1} can be given explicitly by

$$F^{-1}(n_1, n_2) = \frac{(n_1 + n_2 - 2)(n_1 + n_2 - 1)}{2} + n_1, \quad n_1, n_2 \in \mathbb{N}.$$

Thus $\mathbb{N} \times \mathbb{N}$ is a countable set.

Now suppose that $E_1, E_2, \dots$ are countable sets. Then for any $n \in \mathbb{N}$ there exists a bijective mapping $f_n : \mathbb{N} \to E_n$. Let us define a map $g : \mathbb{N} \times \mathbb{N} \to E_1 \cup E_2 \cup \dots$ by $g(n_1, n_2) = f_{n_1}(n_2)$. Obviously, g is onto.

Since the set $\mathbb{N} \times \mathbb{N}$ is countable, there exists a sequence $p_1, p_2, p_3, \dots$ that forms a complete list of its elements. Then the sequence $g(p_1), g(p_2), g(p_3), \dots$ contains all elements of the union $E_1 \cup E_2 \cup E_3 \cup \dots$ Although the latter sequence may include repetitions, we can choose a subsequence $\{g(p_{n_k})\}$ in which every element of the union appears exactly once. Note that the subsequence is infinite since each of the sets $E_1, E_2, \dots$ is infinite.

Now the map $h : \mathbb{N} \to E_1 \cup E_2 \cup E_3 \cup \dots$ defined by $h(k) = g(p_{n_k})$, $k = 1, 2, \dots,$ is a bijection.

Problem 2. Find the following limits:

(i) $\lim_{x\to 0} \log \dfrac{1}{1+\cot(x^2)}$.

The function $f(x) = \log \dfrac{1}{1+\cot(x^2)}$ can be represented as the composition of 4 functions: $f_1(x) = x^2$, $f_2(y) = \cot y$, $f_3(z) = (1+z)^{-1}$, and $f_4(u) = \log u$.

Since the function f_1 is continuous, we have $\lim_{x\to 0} f_1(x) = f_1(0) = 0$. Moreover, $f_1(x) > 0$ for $x \neq 0$. Since $\lim_{y\to 0+} \cot y = +\infty$, it follows that $f_2(f_1(x)) \to +\infty$ as $x \to 0$.

Further, $f_3(z) \to 0+$ as $z \to +\infty$ and $f_4(u) \to -\infty$ as $u \to 0+$. Finally, $f(x) = f_4(f_3(f_2(f_1(x)))) \to -\infty$ as $x \to 0$.

Problem 2. Find the following limits:

(ii) $\lim_{x \to 64} \dfrac{\sqrt{x} - 8}{\sqrt[3]{x} - 4}$.

Consider a function $u(x) = x^{1/6}$ defined on $(0, \infty)$. Since this function is continuous at 64 and $u(64) = 2$, we obtain

$$\lim_{x \to 64} \frac{\sqrt{x} - 8}{\sqrt[3]{x} - 4} = \lim_{x \to 64} \frac{(u(x))^3 - 8}{(u(x))^2 - 4}$$

$$= \lim_{y \to 2} \frac{y^3 - 8}{y^2 - 4} = \lim_{y \to 2} \frac{(y-2)(y^2 + 2y + 4)}{(y-2)(y+2)}$$

$$= \lim_{y \to 2} \frac{y^2 + 2y + 4}{y + 2} = \left. \frac{y^2 + 2y + 4}{y + 2} \right|_{y=2} = 3.$$

Problem 2. Find the following limits:

(iii) $\lim_{n\to\infty}\left(1+\frac{c}{n}\right)^n$, where $c \in \mathbb{R}$.

Let $a_n = (1+c/n)^n$, $n = 1, 2, \ldots$ For n large enough, we have $1 + c/n > 0$ so that $a_n > 0$. Then

$$\log a_n = \log\left(1+\frac{c}{n}\right)^n = n\log\left(1+\frac{c}{n}\right) = \left.\frac{\log(1+cx)}{x}\right|_{x=1/n}.$$

Since $1/n \to 0$ as $n \to \infty$ and

$$\lim_{x\to 0}\frac{\log(1+cx)}{x} = \left.\big(\log(1+cx)\big)'\right|_{x=0} = \left.\frac{c}{1+cx}\right|_{x=0} = c,$$

we obtain that $\log a_n \to c$ as $n \to \infty$. Therefore $a_n = e^{\log a_n} \to e^c$ as $n \to \infty$.

Problem 3. Prove that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

converges to $\sin x$ for any $x \in \mathbb{R}$.

The function $f(x) = \sin x$ is infinitely differentiable on $\mathbb{R}$. According to Taylor's formula, for any $x, x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x, x_0),$$

where $R_n(x, x_0) = \dfrac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)^{n+1}$ for some $\theta = \theta(x, x_0)$ between x and x_0. Since $f'(x) = \cos x$ and $f''(x) = -\sin x = -f(x)$ for all $x \in \mathbb{R}$, it follows that $|f^{(n+1)}(\theta)| \leq 1$ for all $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$. Further, one derives that $R_n(x, x_0) \to 0$ as $n \to \infty$. Thus we obtain an expansion of $\sin x$ into a series. In the case $x_0 = 0$, this is the required series (up to zero terms).

Problem 4. Find an indefinite integral and evaluate definite integrals:

(i) $\displaystyle\int \frac{\sqrt{1+\sqrt[4]{x}}}{2\sqrt{x}}\,dx.$

To find this integral, we change the variable twice. First

$$\int \frac{\sqrt{1+\sqrt[4]{x}}}{2\sqrt{x}}\,dx = \int \sqrt{1+\sqrt[4]{x}}\,(\sqrt{x})'\,dx = \int \sqrt{1+\sqrt{u}}\,du,$$

where $u=\sqrt{x}$. Secondly, we introduce a variable $w=\sqrt{1+\sqrt{u}}$. Then $u=(w^2-1)^2$ so that $du=\big((w^2-1)^2\big)'\,dw = 2(w^2-1)\cdot 2w\,dw = (4w^3-4w)\,dw$. Consequently,

$$\int \sqrt{1+\sqrt{u}}\,du = \int w\,du = \int (4w^4-4w^2)\,dw$$

$$= \frac{4}{5}w^5 - \frac{4}{3}w^3 + C = \frac{4}{5}\big(1+x^{1/4}\big)^{5/2} - \frac{4}{3}\big(1+x^{1/4}\big)^{3/2} + C.$$

Problem 4. Find an indefinite integral and evaluate definite integrals:

(ii) $\displaystyle\int_0^{\sqrt{3}} \frac{x^2+6}{x^2+9}\,dx.$

To evaluate this definite integral, we use linearity of the integral and a substitution $x = 3u$:

$$\int_0^{\sqrt{3}} \frac{x^2+6}{x^2+9}\,dx = \int_0^{\sqrt{3}} \left(1 - \frac{3}{x^2+9}\right)\,dx = \int_0^{\sqrt{3}} 1\,dx$$

$$- \int_0^{\sqrt{3}} \frac{3}{x^2+9}\,dx = \sqrt{3} - \int_0^{\sqrt{3}/3} \frac{3}{(3u)^2+9}\,d(3u)$$

$$= \sqrt{3} - \int_0^{1/\sqrt{3}} \frac{1}{u^2+1}\,du = \sqrt{3} - \arctan u\Big|_{u=0}^{1/\sqrt{3}} = \sqrt{3} - \frac{\pi}{6}.$$

Problem 4. Find an indefinite integral and evaluate definite integrals:

(iii) $\displaystyle\int_0^\infty x^2 e^{-x}\,dx.$

To evaluate the improper integral, we integrate by parts twice:

$$\int_0^\infty x^2 e^{-x}\,dx = -\int_0^\infty x^2 (e^{-x})'\,dx = -\int_0^\infty x^2\,d(e^{-x})$$

$$= -x^2 e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,d(x^2) = \int_0^\infty e^{-x}(x^2)'\,dx$$

$$= \int_0^\infty 2xe^{-x}\,dx = -\int_0^\infty 2x(e^{-x})'\,dx = -\int_0^\infty 2x\,d(e^{-x})$$

$$= -2xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,d(2x) = \int_0^\infty 2e^{-x}\,dx$$

$$= -2e^{-x}\Big|_0^\infty = 2.$$

Problem 5. For each of the following series, determine if the series converges and if it converges absolutely:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$, **(ii)** $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+2^n \cos n}{n!}$, **(iii)** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$.

The first series diverges since

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\left(\sqrt{n+1}+\sqrt{n}\right)^2} > \sum_{n=1}^{\infty} \frac{1}{4(n+1)} = +\infty.$$

The second series can be represented as $\sum_{n=1}^{\infty}(b_n + c_n \cos n)$, where $b_n = \sqrt{n}/n!$ and $c_n = 2^n/n!$ for all $n \in \mathbb{N}$. The series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ both converge (due to the Ratio Test), and so does $\sum_{n=1}^{\infty}(b_n + c_n)$. Since $|b_n + c_n \cos n| \leq b_n + c_n$ for all $n \in \mathbb{N}$, the series $\sum_{n=1}^{\infty}(b_n + c_n \cos n)$ converges absolutely due to the Comparison Test.

Finally, the third series converges (due to the Alternating Series Test), but not absolutely (due to the Integral Test).