MATH 409–501/503 Fall 2013

Sample problems for Test 1

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Prove that for any $n \in \mathbb{N}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Problem 2 (30 pts.) Let $\{F_n\}$ be the sequence of Fibonacci numbers: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(i) Show that the sequence $\{F_{2k}/F_{2k-1}\}_{k\in\mathbb{N}}$ is increasing while the sequence $\{F_{2k+1}/F_{2k}\}_{k\in\mathbb{N}}$ is decreasing.

(ii) Prove that $\displaystyle\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{\sqrt{5}+1}{2}$.

Problem 3 (25 pts.) Prove the Extreme Value Theorem: if $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed bounded interval $[a, b]$, then f is bounded and attains its extreme values (maximum and minimum) on $[a, b]$.

Problem 4 (20 pts.) Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(-1) = f(0) = f(1) = 0$ and $f(x) = \dfrac{x-1}{x^2-1} \sin \dfrac{1}{x}$ for $x \in \mathbb{R} \setminus \{-1, 0, 1\}$.

(i) Determine all points at which the function f is continuous.

(ii) Is the function f uniformly continuous on the interval $(0, 1)$? Is it uniformly continuous on the interval $(1, 2)$? Explain.

Bonus Problem 5 (15 pts.) Given a set X, let $\mathcal{P}(X)$ denote the set of all subsets of X. Prove that $\mathcal{P}(X)$ is not of the same cardinality as X.