MATH 415

Modern Algebra I

Lecture 19:
Review for Exam 2.

Topics for Exam 2

Basic theory of rings and fields:

- Rings and fields
- Integral domains
- Modular arithmetic
- Rings of polynomials
- Factorization of polynomials

Fraleigh: Sections 18–23

Sample problems

Problem 1. Let M be the set of all 2×2 matrices of the form $\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}$, where n and k are rational numbers. Under the operations of matrix addition and multiplication, does this set form a ring? Does M form a field?

Problem 2. Let L be the set of the following 2×2 matrices with entries from the field $\mathbb{Z}_2$:

$$A = \begin{pmatrix} [0] & [0] \\ [0] & [0] \end{pmatrix}, \quad B = \begin{pmatrix} [1] & [0] \\ [0] & [1] \end{pmatrix},$$

$$C = \begin{pmatrix} [1] & [1] \\ [1] & [0] \end{pmatrix}, \quad D = \begin{pmatrix} [0] & [1] \\ [1] & [1] \end{pmatrix}.$$

Under the operations of matrix addition and multiplication, does this set form a ring? Does L form a field?

Sample problems

Problem 3. Prove that for a ring with unity, commutativity of addition follows from the other axioms.

Problem 4. Find a direct product of cyclic groups that is isomorphic to G_{16} (multiplicative group of all invertible elements of the ring $\mathbb{Z}_{16}$).

Problem 5. Determine the last two digits of 303^{303}.

Problem 6. Find all integer solutions of the equation $21x - 32y = 4$.

Sample problems

Problem 7. Find all integer solutions of the equation $2x + 3y + 5z = 7$.

Problem 8. Solve the equation $2x^{100} + x^{71} + x^{29} = 0$ over the field $\mathbb{Z}_{11}$.

Problem 9. Factor a polynomial $p(x) = x^3 - 3x^2 + 3x - 2$ into irreducible factors over the field $\mathbb{Z}_7$.

Problem 10. Factor a polynomial $p(x) = x^4 + x^3 - 2x^2 + 3x - 1$ into irreducible factors over the field $\mathbb{Q}$.

Problem 1. Let M be the set of all 2×2 matrices of the form $\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}$, where n and k are rational numbers. Under the operations of matrix addition and multiplication, does this set form a ring? Does M form a field?

The set M is closed under matrix addition, taking the negative, and matrix multiplication as

$$\begin{pmatrix} n & k \\ 0 & n \end{pmatrix} + \begin{pmatrix} n' & k' \\ 0 & n' \end{pmatrix} = \begin{pmatrix} n+n' & k+k' \\ 0 & n+n' \end{pmatrix},$$

$$-\begin{pmatrix} n & k \\ 0 & n \end{pmatrix} = \begin{pmatrix} -n & -k \\ 0 & -n \end{pmatrix},$$

$$\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}\begin{pmatrix} n' & k' \\ 0 & n' \end{pmatrix} = \begin{pmatrix} nn' & nk'+kn' \\ 0 & nn' \end{pmatrix}.$$

Also, the multiplication is commutative on M. The associativity and commutativity of the addition, the associativity of the multiplication, and the distributive law hold on M since they hold for all 2×2 matrices. Thus M is a commutative ring.

Problem 1. Let M be the set of all 2×2 matrices of the form $\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}$, where n and k are rational numbers. Under the operations of matrix addition and multiplication, does this set form a ring? Does M form a field?

The ring M is not a field since it has zero-divisors
(and zero-divisors do not admit multiplicative inverses).
For example, the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M$ is a divisor of zero as

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Problem 2. Let L be the set of the following 2×2 matrices with entries from the field $\mathbb{Z}_2$:

$$A = \begin{pmatrix} [0] & [0] \\ [0] & [0] \end{pmatrix}, \ B = \begin{pmatrix} [1] & [0] \\ [0] & [1] \end{pmatrix}, \ C = \begin{pmatrix} [1] & [1] \\ [1] & [0] \end{pmatrix}, \ D = \begin{pmatrix} [0] & [1] \\ [1] & [1] \end{pmatrix}.$$

Under the operations of matrix addition and multiplication, does this set form a ring? Does L form a field?

First we build the addition and multiplication tables for L (meanwhile checking that L is closed under both operations):

$+$	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

$\times$	A	B	C	D
A	A	A	A	A
B	A	B	C	D
C	A	C	D	B
D	A	D	B	C

Analyzing these tables, we find that both operations are commutative on L, A is the additive identity element, and B is the multiplicative identity element. Also, $B^{-1} = B$, $C^{-1} = D$, $D^{-1} = C$, and $-X = X$ for all $X \in L$. The associativity of addition and multiplication as well as the distributive law hold on L since they hold for all 2×2 matrices. Thus L is a field.

Problem 3. Prove that for a ring with unity, commutativity of addition follows from the other axioms.

Suppose R is a set with two operations, addition and multiplication, that satisfies all axioms of a ring with unity except, possibly, commutativity of addition. We need to show that addition is commutative anyway: $x+y=y+x$ for all $x,y\in R$. Let us simplify $(1+1)(x+y)$ in two different ways:

$$(1+1)(x+y)=1(x+y)+1(x+y)=(x+y)+(x+y),$$

$$\begin{aligned}(1+1)(x+y)&=(1+1)x+(1+1)y\\&=(1x+1x)+(1y+1y)=(x+x)+(y+y).\end{aligned}$$

Hence $(x+y)+(x+y)=(x+x)+(y+y)$. It follows that

$$(-x)+(x+y)+(x+y)+(-y)=(-x)+(x+x)+(y+y)+(-y),$$

$$(-x+x)+(y+x)+(y+(-y))=(-x+x)+(x+y)+(y+(-y)),$$

$$0+(y+x)+0=0+(x+y)+0 \implies y+x=x+y.$$

Remark. The same argument proves that for a vector space, commutativity of vector addition follows from the other axioms.

Problem 4. Find a direct product of cyclic groups that is isomorphic to G_{16} (multiplicative group of all invertible elements of the ring $\mathbb{Z}_{16}$).

A congruence class $[a]_{16}$ is invertible in $\mathbb{Z}_{16}$ if and only if a is coprime with 16, that is, if a is odd. There are 8 congruence classes in G_{16}: $[1], [3], [5], [7], [9], [11], [13], [15]$.

Classification of finite abelian groups implies that G_{16} is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. These three groups are distinguished by orders of their elements: $\mathbb{Z}_8$ has elements of order 1, 2, 4 and 8; $\mathbb{Z}_4 \times \mathbb{Z}_2$ has elements of order 1, 2 and 4; $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has only elements of order 1 and 2.

Let us find orders for all elements of G_{16}.

$[1]$ has order 1.

$[3]^2 = [9],\ [3]^4 = [9]^2 = [81] = [1],$ hence $[3]$ has order 4.

$[5]^2 = [25] = [9],\ [5]^4 = [9]^2 = [1],$ hence $[5]$ has order 4.

$[7]^2 = [49] = [1],$ hence $[7]$ has order 2.

$[9]^2 = [1],$ hence $[9]$ has order 2.

$[11]^2 = [-5]^2 = [5]^2 = [9],$ hence $[11]$ has order 4.

$[13]^2 = [-3]^2 = [9],$ hence $[13]$ has order 4.

$[15]^2 = [-1]^2 = [1],$ hence $[15]$ has order 2.

We conclude that $G_{16} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

Problem 5. Determine the last two digits of 303^{303}.

The last two digits form the remainder under division by 100. We know that $\phi(100) = 40$. It follows from Euler's Theorem that $3^{40} \equiv 1 \bmod 100$. Then

$$[303^{303}] = [303]^{303} = [3]^{303} = [3]^{40\cdot 7+23} = ([3]^{40})^7\,[3]^{23} = [3]^{23}.$$

We have $[3]^2 = [9]$, $[3]^3 = [9][3] = [27]$, $[3]^4 = [27][3] = [81]$,
$[3]^5 = [81][3] = [43]$, $[3]^6 = [43][3] = [29]$,
$[3]^7 = [29][3] = [87]$, $[3]^8 = [87][3] = [61]$,
$[3]^9 = [61][3] = [83]$, $[3]^{10} = [83][3] = [49]$,
$[3]^{11} = [49][3] = [47]$, $[3]^{12} = [47][3] = [41]$,
$[3]^{13} = [41][3] = [23]$, $[3]^{14} = [23][3] = [69]$,
$[3]^{15} = [69][3] = [7]$, $[3]^{16} = [7][3] = [21]$,
$[3]^{17} = [21][3] = [63]$, $[3]^{18} = [63][3] = [89]$,
$[3]^{19} = [89][3] = [67]$, $[3]^{20} = [67][3] = [1]$,
Finally, $[3]^{23} = [3]^3 = [27]$ so that $303^{303} = \ldots 27$.

Remark. It turns out that $G_{100} \cong \mathbb{Z}_{20} \times \mathbb{Z}_2$. Therefore the order of each element of the group G_{100} is a divisor of 20.

Problem 5. Determine the last two digits of 303^{303}.

Alternative solution: The last two digits form the remainder under division by 100. First let us find the remainders under division by 25 and 4. We have $\phi(25) = 25 - 5 = 20$ and $\phi(4) = 4 - 2 = 2$. It follows from Euler's Theorem that $303^{20} \equiv 1 \bmod 25$ and $303^2 \equiv 1 \bmod 4$. Then

$$\begin{aligned}
[303^{303}]_{25} &= [303]_{25}^{303} = [303]_{25}^{20\cdot 15+3} = ([303]_{25}^{20})^{15}\,[303]_{25}^{3} \\
&= [303]_{25}^{3} = [3]_{25}^{3} = [3^3]_{25} = [27]_{25} = [2]_{25}, \\
[303^{303}]_{4} &= [303]_{4}^{303} = [303]_{4}^{2\cdot 151+1} = ([303]_{4}^{2})^{151}\,[303]_{4} \\
&= [303]_{4} = [3]_{4}.
\end{aligned}$$

Since $303^{303} \equiv 2 \bmod 25$, the remainder of 303^{303} under division by 100 is among the four numbers 2, $27 = 2 + 25$, $52 = 2 + 25 \cdot 2$, and $77 = 2 + 25 \cdot 3$. We pick the one that has remainder 3 under division by 4. That's 27.

Problem 6. Find all integer solutions of the equation $21x - 32y = 4$.

An integer y is a part of an integer solution (x, y) of the equation if and only if it is a solution of the linear congruence $-32y \equiv 4 \bmod 21$. Since $-32 \equiv 10 \bmod 21$, this is equivalent to $10y \equiv 4 \bmod 21$. Further, we can cancel the common factor 2 on both sides of the congruence (since 2 is coprime with 21): $10y \equiv 4 \bmod 21 \iff 5y \equiv 2 \bmod 21$. To solve the latter linear congruence, we need to find the multiplicative inverse of 5 modulo 21. This is -4 as $-4 \cdot 5 = -20 \equiv 1 \bmod 21$. Hence

$$5y \equiv 2 \bmod 21 \iff y \equiv -4 \cdot 2 \equiv -8 \bmod 21.$$

In other words, $y = -8 + 21k$ for some $k \in \mathbb{Z}$. The corresponding value of x can be found from the equation: $x = (4 + 32y)/21 = (4 + 32(-8 + 21k))/21 = -12 + 32k$ (it should be integer as well). Thus the general integer solution is $x = -12 + 32k$, $y = -8 + 21k$, where $k \in \mathbb{Z}$.

Problem 7. Find all integer solutions of the equation $2x + 3y + 5z = 7$.

Let us rewrite the equation as $2x + 3y = c(z)$, where $c(z) = 7 - 5z$, and consider $c(z)$ an integer parameter.

If (x, y) is an integer solution, then x is a solution of the congruence $2x \equiv c(z) \bmod 3$. Then $4x \equiv 2c(z) \bmod 3$ and $x \equiv 2c(z) \bmod 3$. Conversely, if $x = 2c(z) + 3k$, where $k \in \mathbb{Z}$, then we can find y from the equation, $y = (c(z) - 2x)/3 = -c(z) - 2k$, and it is also an integer. All this can be done for any integer value of z.

Thus the general integer solution of the original equation is

$$z = m,$$
$$x = 2c(m) + 3k = 3k - 10m + 14,$$
$$y = -c(m) - 2k = -2k + 5m - 7,$$

where k and m are arbitrary integers.

Problem 8. Solve the equation $2x^{100} + x^{71} + x^{29} = 0$ over the field $\mathbb{Z}_{11}$.

The equation is equivalent to

$$x^{29}(2x^{71} + x^{42} + 1) = 0.$$

Hence $x = 0$ or $2x^{71} + x^{42} + 1 = 0$. By Fermat's Little Theorem, $x^{10} = 1$ for any nonzero $x \in \mathbb{Z}_{11}$. Since 0 is not a solution of the equation $2x^{71} + x^{42} + 1 = 0$, this equation is equivalent to $2x + x^2 + 1 = 0 \iff (x+1)^2 = 0 \iff x = -1$.

Thus the solutions are $x = 0$ and $x = 10$ (note that $-1 \equiv 10 \bmod 11$).

Problem 9. Factor a polynomial $p(x) = x^3 - 3x^2 + 3x - 2$ into irreducible factors over the field $\mathbb{Z}_7$.

A quadratic or cubic polynomial is irreducible if and only if it has no zeros. Indeed, if such a polynomial splits into a product of two non-constant polynomials, then at least one of the factors is linear. This implies that the original polynomial has a zero.

Let us look for the zeros of $p(x)$: $p(0) = -2$, $p(1) = -1$, $p(2) = 0$. Hence $p(x)$ is divisible by $x - 2$:

$$x^3 - 3x^2 + 3x - 2 = (x - 2)(x^2 - x + 1).$$

Now let us look for the zeros of the polynomial $q(x) = x^2 - x + 1$. Note that values 0 and 1 can be skipped this time. We obtain $q(2) = 3$, $q(3) = 7 \equiv 0 \bmod 7$. Hence $q(x)$ is divisible by $x - 3$: $x^2 - x + 1 = (x - 3)(x + 2)$.

Thus $x^3 - 3x^2 + 3x - 2 = (x - 2)(x - 3)(x + 2)$ over the field $\mathbb{Z}_7$.

Problem 10. Factor $p(x) = x^4 + x^3 - 2x^2 + 3x - 1$ into irreducible factors over the field $\mathbb{Q}$.

Possible rational zeros of p are 1 and -1. They are not zeros. Hence p is either irreducible over $\mathbb{Q}$ or else it is factored as

$$x^4 + x^3 - 2x^2 + 3x - 1 = (ax^2 + bx + c)(a'x^2 + b'x + c').$$

Since $p \in \mathbb{Z}[x]$, one can show that the factorization (if it exists) can be chosen so that all coefficients are integer. Additionally, we can assume that $a \geq 0$ (otherwise we could multiply each factor by -1). Equating the corresponding coefficients of the left-hand side and the right-hand side, we obtain $aa' = 1$, $ab' + a'b = 1$, $ac' + bb' + a'c = -2$, $bc' + b'c = 3$ and $cc' = -1$. The first and the last equations imply that $a = a' = 1$, $c = 1$ or -1, and $c' = -c$. Then $b + b' = 1$ and $bb' = -2$, which implies $\{b, b'\} = \{2, -1\}$. Finally, $c = -1$ if $b = 2$ and $c = 1$ if $b = -1$. We can check that indeed

$$x^4 + x^3 - 2x^2 + 3x - 1 = (x^2 + 2x - 1)(x^2 - x + 1).$$