MATH 614

Dynamical Systems and Chaos

Lecture 1:
Examples of dynamical systems.

A **discrete dynamical system** is simply a transformation $f : X \to X$. The set X is regarded the phase space of the system and the map f is considered the law of evolution over a period of time. Given an initial point $x_0 \in X$, the theory of dynamical systems is concerned with asymptotic behavior of a sequence $x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \dots,$ which is called the **orbit** of the point x_0. There are several questions to address here:

- behavior of an individual orbit (say, is it periodic?);
- global behavior of the system (say, are there interesting invariant sets?);
- what happens when we perturb x_0 (is the system regular or chaotic?);
- what happens when we perturb f (is the system structurally stable?).

A **continuous dynamical system** (or a **flow**) is a one-parameter family of maps $T^t : X \to X$, $t > 0$, such that $T^t \circ T^s = T^{t+s}$ for all $t, s > 0$.

Example of a flow

Consider an autonomous system of n ordinary differential equations of the first order

$$\begin{cases} \dot{x}_1 = g_1(x_1, x_2, \dots, x_n), \\ \dot{x}_2 = g_2(x_1, x_2, \dots, x_n), \\ \dots\dots\dots \\ \dot{x}_n = g_n(x_1, x_2, \dots, x_n), \end{cases}$$

where $g_1, g_2, \dots, g_n$ are differentiable functions defined in a domain $D \subset \mathbb{R}^n$. In vector form, $\dot{\mathbf{v}} = G(\mathbf{v})$, where $G : D \to \mathbb{R}^n$ is a vector field. Assume that for any $\mathbf{x} \in D$ the initial value problem $\dot{\mathbf{v}} = G(\mathbf{v})$, $\mathbf{v}(0) = \mathbf{x}$ has a unique solution $\mathbf{v}_\mathbf{x}(t)$, $t \geq 0$. Then the system of ODEs gives rise to a dynamical system with continuous time $F^t : D \to D$, $t \geq 0$ defined by $F^t(\mathbf{x}) = \mathbf{v}_\mathbf{x}(t)$ for all $\mathbf{x} \in D$ and $t \geq 0$.

In the case G is linear, $G(\mathbf{v}) = A\mathbf{v}$ for some $n \times n$ matrix A, the flow is also linear, $F^t(\mathbf{x}) = e^{tA}\mathbf{x}$.

The first return map

Suppose $f : X \to X$ is a discrete dynamical system and X_0 is a subset of the phase space X.

Definition. The **first return map** (or **Poincare map**) of f on X_0 is a map $f_0 : X_0 \to X_0$ defined by

$$f_0(x) = f^{n(x)}(x), \quad x \in X_0,$$

where $n(x)$ is the least positive integer n such that $f^n(x) \in X_0$.

Note that f_0 might not be well defined on the entire set X_0.

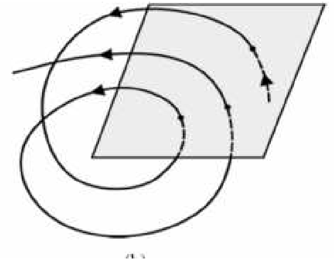
The first return map can be used to study the dynamical system using renormalization techniques.

The first return map

Similarly, given a continuous dynamical system $T^t : X \to X$ and a subset $X_0 \subset X$, we can define the **first return map** $f_0 : X_0 \to X_0$ of the flow T^t by

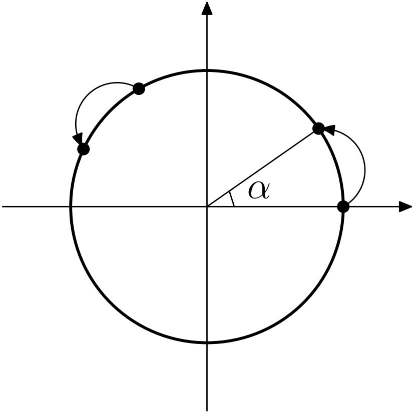
$$f_0(x) = T^{t(x)}(x), \ \ x \in X_0,$$

where $t(x)$ is the least number $t > 0$ such that $T^t(x) \in X_0$.



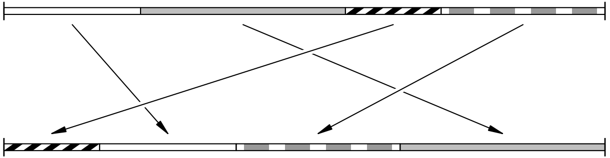
Again, f_0 might not be well defined on the entire set X_0.
For a continuous dynamical system, the first return map often allows to reduce the dimension of the phase space by 1.

Rotation of the circle



$R_\alpha : S^1 \to S^1$, rotation by angle $\alpha \in \mathbb{R}$.
All rotations R_α, $\alpha \in \mathbb{R}$ form a flow on S^1.

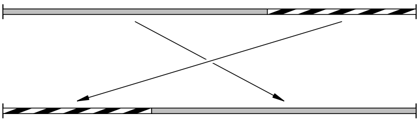
Interval exchange transformation



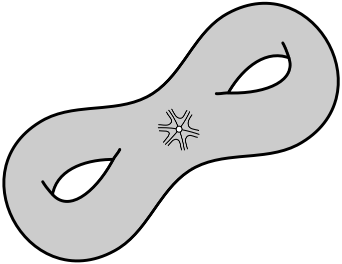
An **interval exchange transformation** of an interval I is defined by cutting the interval into several subintervals and then rearranging them by translation.

Combinatorial description: (λ, π), where
$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$, $\lambda_i > 0$, $\lambda_1 + \cdots + \lambda_n = |I|$;
π is a permutation on $\{1, 2, \ldots n\}$.
In the example, $\pi = (1\,2\,4\,3)$.

The exchange of two intervals is equivalent to a rotation of the circle.



Interval exchange transformations arise as the first return maps for certain flows on surfaces.

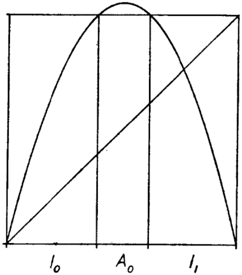


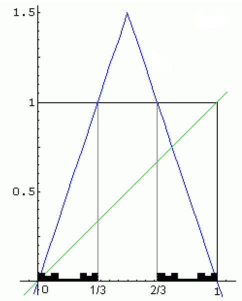
Unimodal maps

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map such that

- $f(0) = f(1) = 0$;
- there exists a point $x_{\max} \in (0, 1)$ such that f is strictly increasing on $(-\infty, x_{\max}]$ and strictly decreasing on $[x_{\max}, \infty)$;

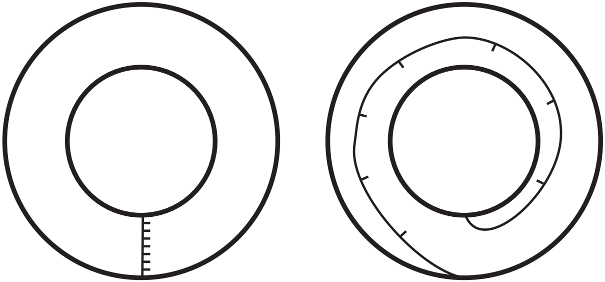
The map f is called **unimodal**.





Twist map

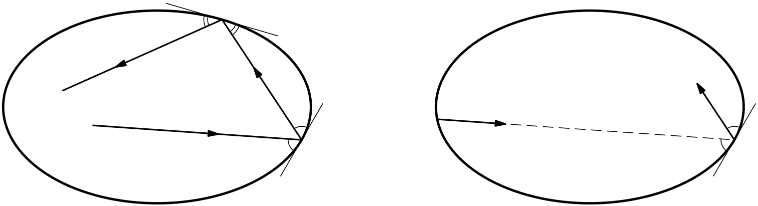
A **twist map** is a homeomorphism of an annulus that fixes both boundary circles (pointwise!) but rotates them relative to each other.



Example. U is an annulus given by $1 \leq r \leq 2$ in polar coordinates (r, ϕ). A twist map $T : U \to U$ is defined by $T(r, \phi) = (r, \phi + 2\pi(r - 1))$.

The annulus is foliated by invariant circles (rotated by T).

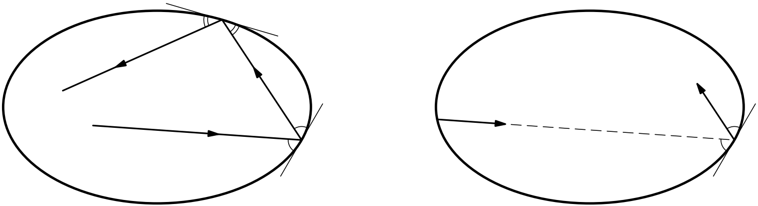
Billiard



D: a bounded domain with piecewise smooth boundary in $\mathbb{R}^2$ (a billiard table).

The **billiard flow** in D is a dynamical system describing uniform motion with unit speed inside D of a point representing the billiard ball and with reflections off the boundary according to the law *the angle of incidence is equal to the angle of reflection*. The phase space of the flow is $D \times S^1$ (unit tangent bundle) up to some identifications on the boundary.

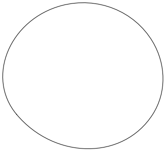
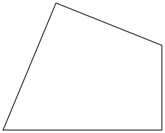
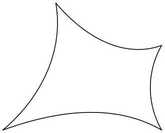
Billiard



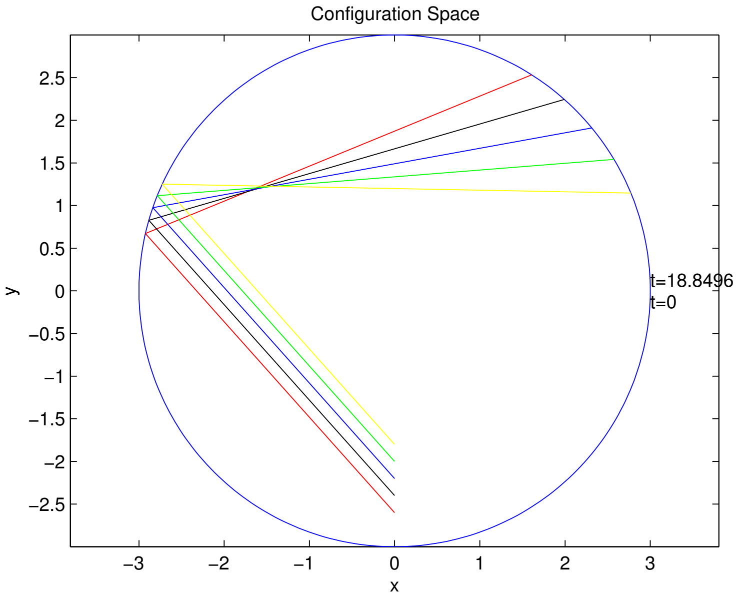
The **billiard ball map** of $\partial D \times S^1$ (modulo identifications) is a first-return map of the billiard flow.

In the case the billiard table D is convex and smooth, the billiard ball map can be represented as a twist map.

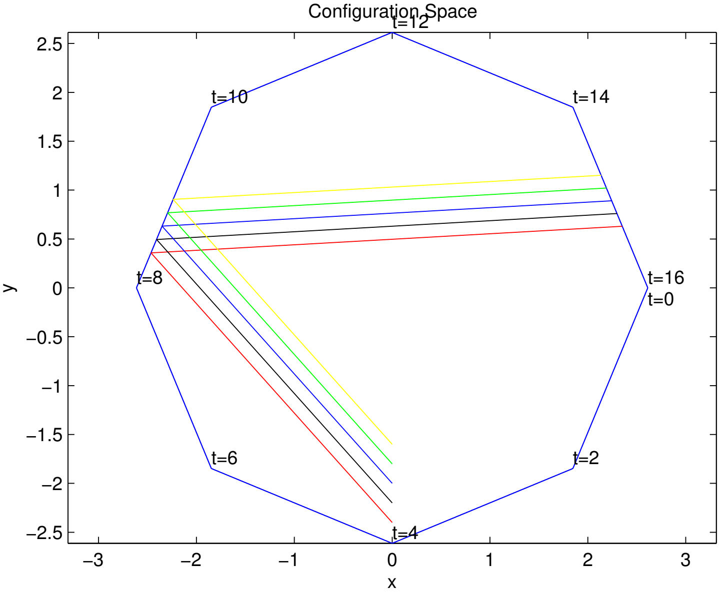
Three types of boundary

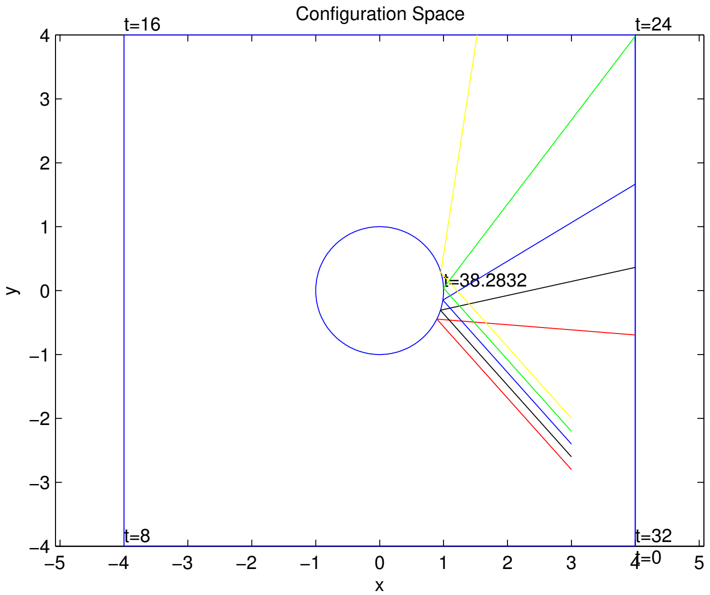
Birkhoff billiards	polygonal billiards	Sinai billiards
regular	intermediate	chaotic
focusing	neutral	dispersing



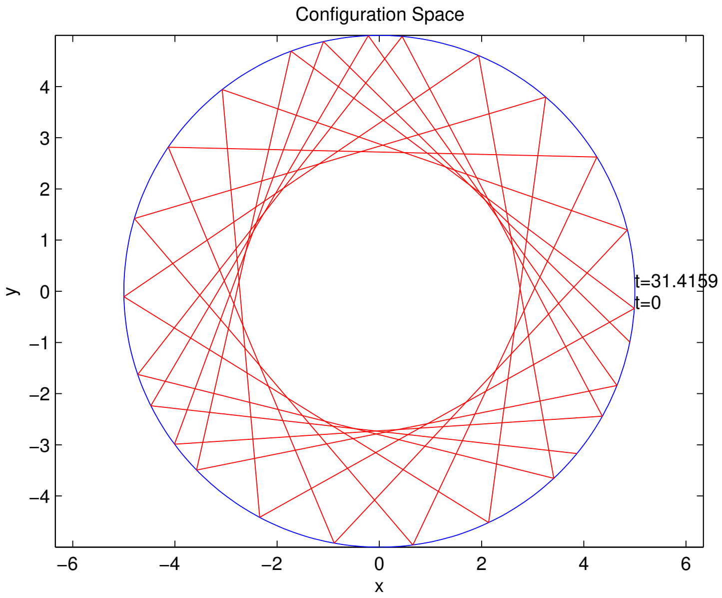
Configuration Space
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
y
−3
−2
−1
0
1
2
3
x
t=18.8496
t=0



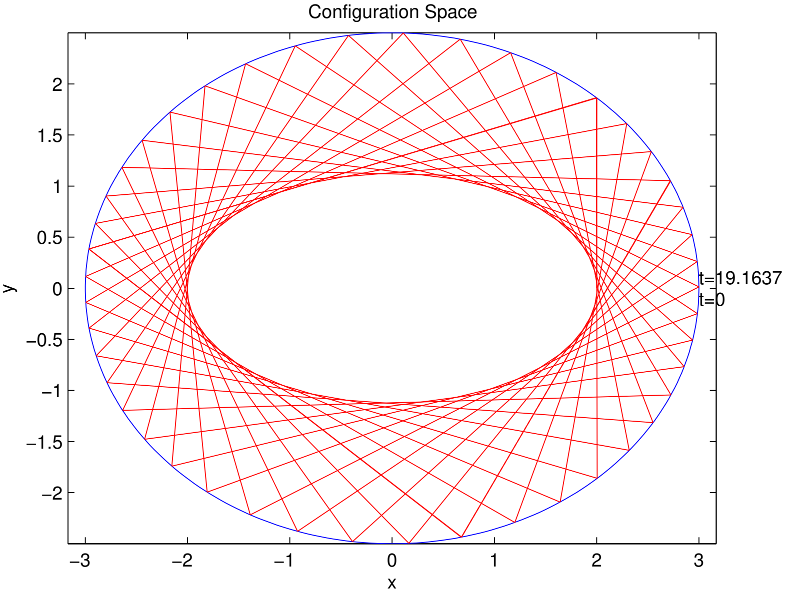
Configuration Space
t=12
t=10
t=14
t=8
t=16
t=0
t=6
t=2
t=4
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−2
−1
0
1
2
3
x
y

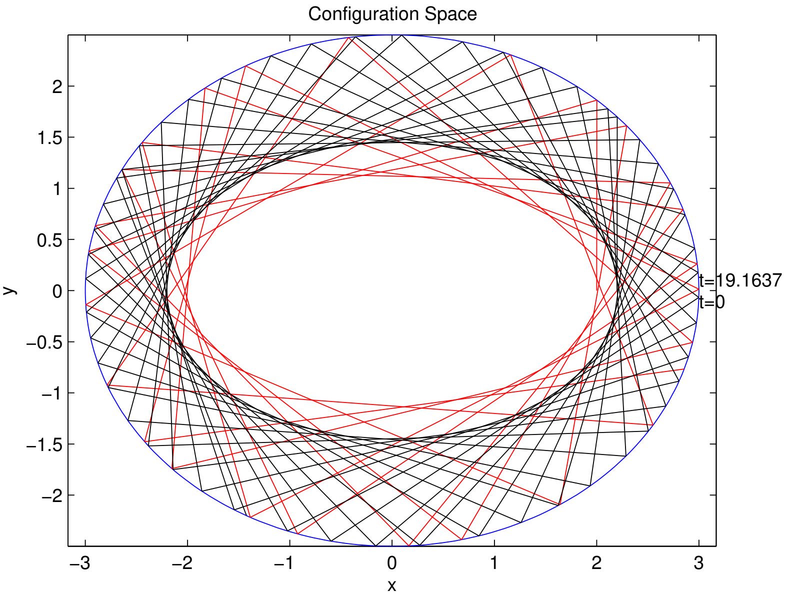

Configuration Space
t=16
t=24
t=38.2832
t=8
t=32
t=0
x
y
4
3
2
1
0
−1
−2
−3
−4
−5
−4
−3
−2
−1
0
1
2
3
4
5

Billiard in a circle

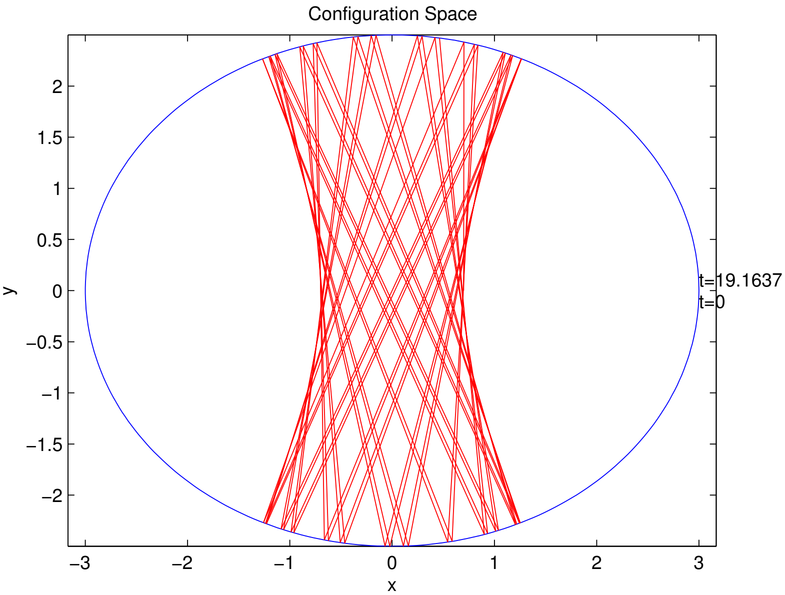


Billiard in an ellipse

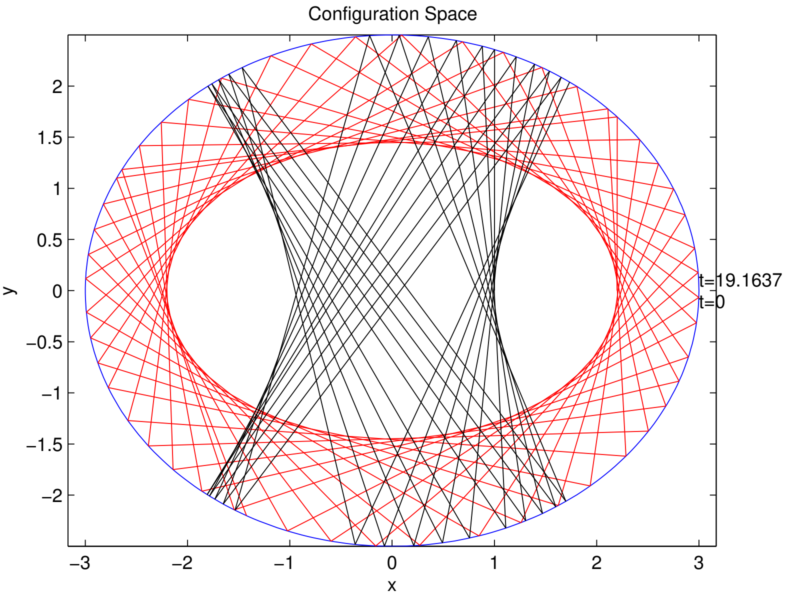




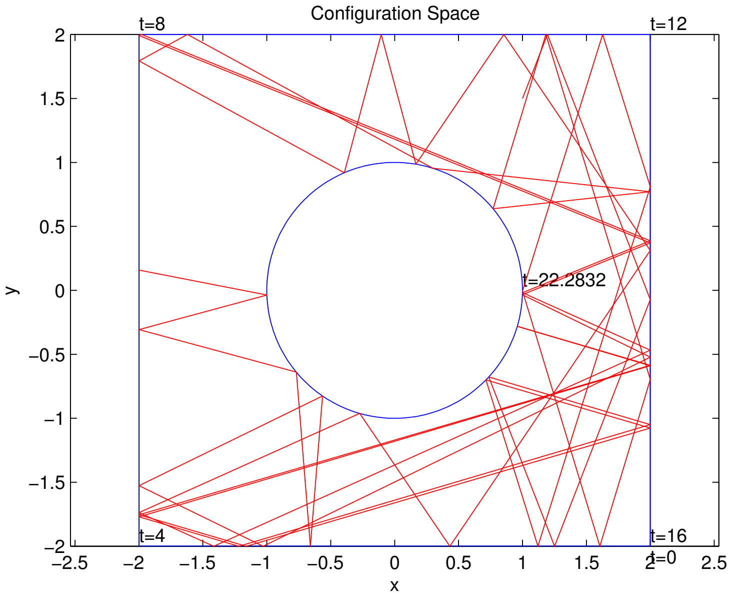
Configuration Space
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
y
−3
−2
−1
0
1
2
3
x
t=19.1637
t=0



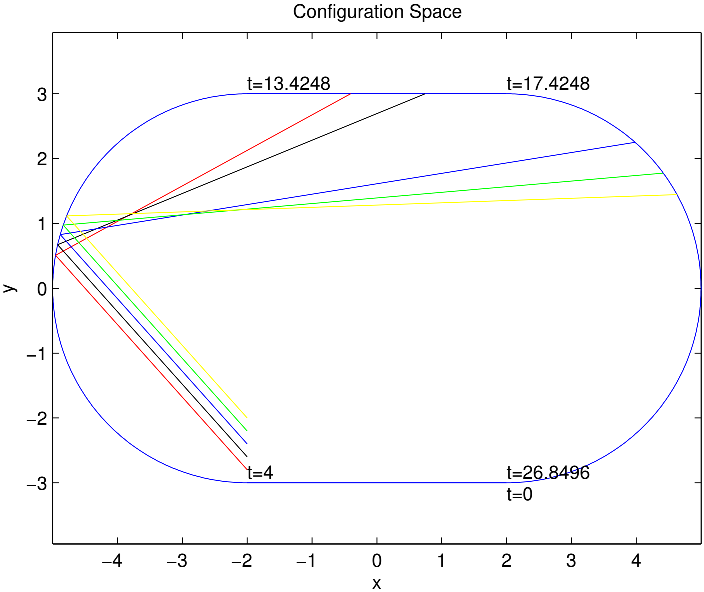
Configuration Space
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
y
−3
−2
−1
0
1
2
3
x
t=19.1637
t=0


Configuration Space
t=19.1637
t=0
x
y
−3
−2
−1
0
1
2
3
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2

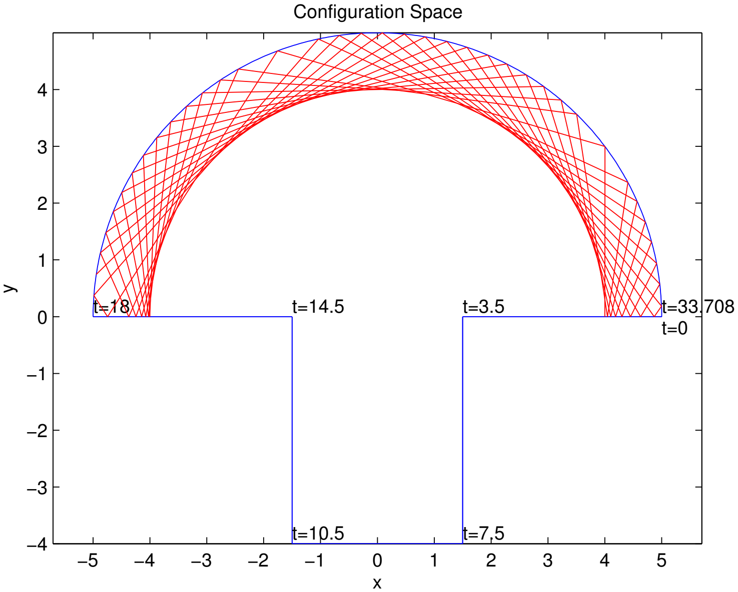
Sinai billiard

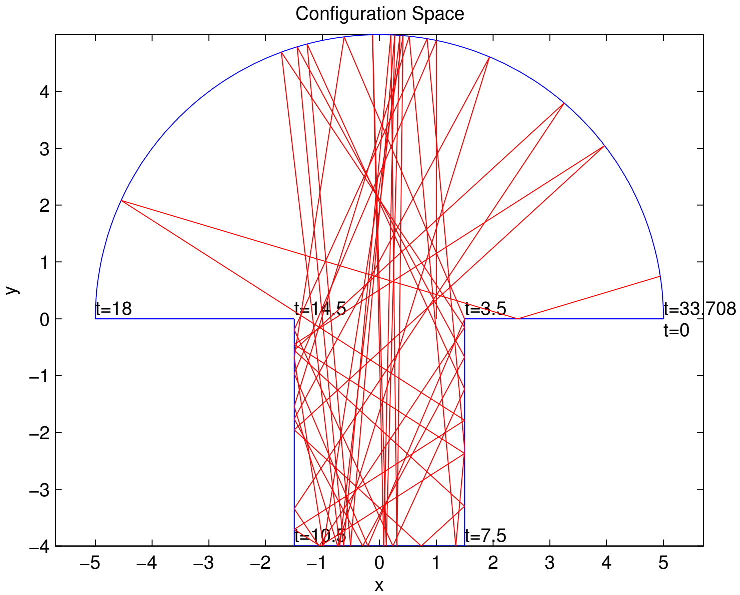


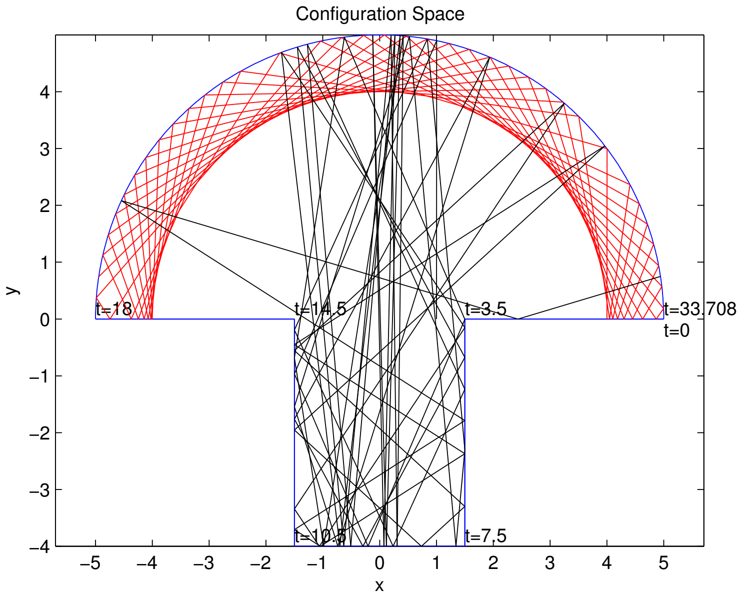
Stadium billiard



Mushroom billiard




Configuration Space
t=18
t=14.5
t=3.5
t=33.708
t=0
t=10.5
t=7.5
y
x
4
3
2
1
0
−1
−2
−3
−4
−5
5



Configuration Space
y
x
t=18
t=14.5
t=3.5
t=33.708
t=0
t=10.5
t=7.5