MATH 614

Dynamical Systems and Chaos

Lecture 7:
Symbolic dynamics (continued).

Symbolic dynamics

Given a finite set $\mathcal{A}$ (an alphabet), we denote by $\Sigma_{\mathcal{A}}$ the set of all infinite words over $\mathcal{A}$, i.e., infinite sequences $\mathbf{s} = (s_1 s_2 \dots)$, $s_i \in \mathcal{A}$.

For any finite word w over the alphabet $\mathcal{A}$, that is, $w = s_1 s_2 \dots s_n$, $s_i \in \mathcal{A}$, we define a **cylinder** $C(w)$ to be the set of all infinite words $\mathbf{s} \in \Sigma_{\mathcal{A}}$ that begin with w. The topology on $\Sigma_{\mathcal{A}}$ is defined so that open sets are unions of cylinders. Two infinite words are considered close in this topology if they have a long common beginning.

The **shift** transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is defined by $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 \dots)$. This transformation is continuous. The study of the shift and related transformations is called **symbolic dynamics**.

Properties of the shift

- The shift transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is continuous.

- An infinite word $\mathbf{s} \in \Sigma_{\mathcal{A}}$ is a periodic point of the shift if and only if $\mathbf{s} = www\ldots$ for some finite word w.

- An infinite word $\mathbf{s} \in \Sigma_{\mathcal{A}}$ is an eventually periodic point of the shift if and only if $\mathbf{s} = uwww\ldots$ for some finite words u and w.

- The shift σ has periodic points of all (prime) periods.

Dense sets

Definition. Suppose (X, d) is a metric space. We say that a subset $E \subset X$ is **everywhere dense** (or simply **dense**) in X if for every $x \in X$ and $\varepsilon > 0$ there exists $y \in E$ such that $d(y, x) < \varepsilon$.

More generally, suppose X is a topological space. We say that a subset $E \subset X$ is **dense** in X if E intersects every nonempty open subset of X.

Proposition Periodic points of the shift $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ are dense in $\Sigma_{\mathcal{A}}$.

Proof: Let w be any nonempty finite word over the alhabet $\mathcal{A}$. Then the cylinder $C(w)$ contains a periodic point, e.g., $www\dots$ Consequently, any nonempty open set $U \subset \Sigma_{\mathcal{A}}$ contains a periodic point.

Dense orbit of the shift

Proposition The shift transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ admits a dense orbit.

Proof: Since open subsets of $\Sigma_{\mathcal{A}}$ are unions of cylinders, it follows that a set $E \subset \Sigma_{\mathcal{A}}$ is dense if and only if it intersects every cylinder.

The orbit under the shift of an infinite word $\mathbf{s} \in \Sigma_{\mathcal{A}}$ visits a particular cylinder $C(w)$ if and only if the finite word w appears somewhere in $\mathbf{s}$, that is, $\mathbf{s} = w_0 w \mathbf{s}_0$, where w_0 is a finite word and $\mathbf{s}_0$ is an infinite word. Therefore the orbit $O_\sigma^+(\mathbf{s})$ is dense in $\Sigma_{\mathcal{A}}$ if and only if the infinite word $\mathbf{s}$ contains all finite words over the alphabet $\mathcal{A}$ as subwords.

There are only countably many finite words over $\mathcal{A}$. We can enumerate them all: $w_1, w_2, w_3, \dots$ Then an infinite word $\mathbf{s} = w_1 w_2 w_3 \dots$ has dense orbit.

Applications of symbolic dynamics

Suppose $f : X \to X$ is a dynamical system. Given a partition of the set X into disjoint subsets X_α, $\alpha \in \mathcal{A}$ indexed by elements of a finite set $\mathcal{A}$, we can define the **itinerary map** $S : X \to \Sigma_{\mathcal{A}}$ so that $S(x) = (s_0 s_1 s_2 \dots)$, where $f^n(x) \in X_{s_n}$ for all $n \geq 0$.

In the case f is continuous, the itinerary map is continuous if the sets X_α are **clopen** (i.e., both closed and open).

Indeed, for any finite word $w = s_0 s_1 \dots s_k$ over the alphabet $\mathcal{A}$ the preimage of the cylinder $C(w)$ under the itinerary map is

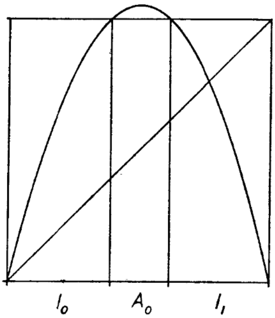
$$S^{-1}(C(w)) = X_{s_0} \cap f^{-1}(X_{s_1}) \cap \cdots \cap (f^k)^{-1}(X_{s_k}).$$

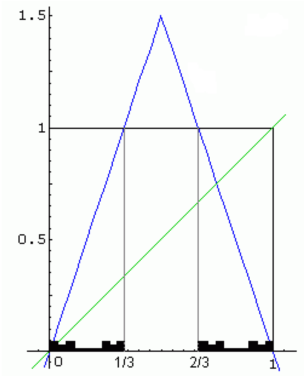
Applications of symbolic dynamics

A more general construction is to take disjoint open sets $X_\alpha,\ \alpha \in \mathcal{A}$ that need not cover the entire set X. Then the itinerary map is defined on a subset of X consisting of all points whose orbits stay in the union of the sets X_α.

Alternatively, we can consider a partition into sets that are not open (but then the itinerary map will not be continuous at some points). Alternatively, we can allow the sets X_α to overlap (but then the itinerary map will not be uniquely defined at some points).

Examples





Any real number x is uniquely represented as $x = k + r$, where $k \in \mathbb{Z}$ and and $0 \leq r < 1$. Then k is called the **integer part** of x and r is called the **fractional part** of x. Notation: $k = [x]$, $r = \{x\}$.

Example. $f : [0, 1) \to [0, 1)$, $f(x) = \{10x\}$.

Consider a partition of the interval $[0, 1)$ into 10 subintervals $X_i = [\frac{i}{10}, \frac{i+1}{10})$, $0 \leq i \leq 9$. That is, $X_0 = [0, 0.1)$, $X_1 = [0.1, 0.2)$,..., $X_9 = [0.9, 1)$.

Given a point $x \in [0, 1)$, let $S(x) = (s_0 s_1 s_2 \dots)$ be the itinerary of x relative to that partition. Then $0.s_0 s_1 s_2 \dots$ is the decimal expansion of the real number x.

Totally disconnected sets

Let X be a topological space and $E \subset X$. We say that points $x, y \in E$ are **disconnected** in E if there exist disjoint open sets $U_x, U_y \subset X$ such that $x \in U_x$, $y \in U_y$, and $E \subset U_x \cup U_y$. The set E is called **connected** if no points in E are disconnected. The set E is called **totally disconnected** if any two points of E are disconnected.

Suppose (X, d) is a metric space. The space X is called **ultrametric** (or **non-Archimedean**) if $d(x, y) \leq \max(d(x, z), d(z, y))$ for all $x, y, z \in X$.

Theorem Any ultrametric space is totally disconnected.

Idea of the proof: In the ultrametric space, two balls $B_\varepsilon(x)$ and $B_\varepsilon(y)$ of the same radius are either disjoint or the same.

Theorem The space $\Sigma_{\mathcal{A}}$ of infinite sequences is ultrametric (and hence totally disconnected).