MATH 614

Dynamical Systems and Chaos

Lecture 10:
Chaos in unimodal maps.
Structural stability.

Definition of chaos

Suppose $f : X \to X$ is a continuous transformation of a metric space (X, d).

Definition. We say that the map f is **chaotic** if

- f has sensitive dependence on initial conditions;
- f is topologically transitive;
- periodic points of f are dense in X.

Theorem For continuous transformations of compact metric spaces, chaoticity is preserved under topological conjugacy.

Separation of orbits

Suppose $f : X \to X$ is a continuous transformation of a metric space (X, d).

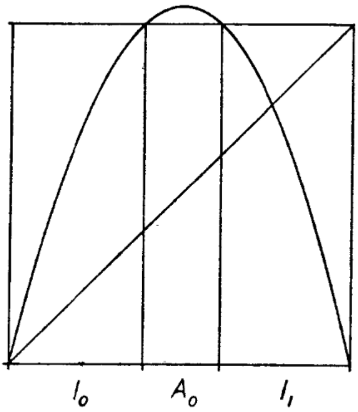
Definition. We say that f has **sensitive dependence on initial conditions** if there is $\delta > 0$ such that, for any $x \in X$ and a neighborhood U of x, there exist $y \in U$ and $n \geq 0$ satisfying $d\big(f^n(y), f^n(x)\big) > \delta$.

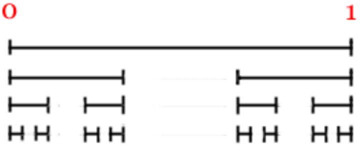
We say that the map f is **expansive** if there is $\delta > 0$ such that, for any $x, y \in X$, $x \neq y$, there exists $n \geq 0$ satisfying $d\big(f^n(y), f^n(x)\big) > \delta$.

Proposition If X is compact, then changing the metric d to another metric that induces the same topology cannot affect sensitive dependence on i.c. and expansiveness of the map f.

Corollary For continuous transformations of compact metric spaces, sensitive dependence on initial conditions and expansiveness are preserved under topological conjugacy.

Unimodal maps





Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map, Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0,1]$, and $S : \Lambda \to \Sigma_{\{0,1\}}$ be the itinerary map. We know that S is a continuous semi-conjugacy of the restriction $f|_\Lambda$ with the shift.

Theorem The following conditions are equivalent:

(i) Λ is a Cantor set;

(ii) the itinerary map S is one-to-one;

(iii) the restriction $f|_\Lambda$ is topologically conjugate to the shift;

(iv) $f|_\Lambda$ has sensitive dependence on initial conditions;

(v) the restriction $f|_\Lambda$ is expansive;

(vi) the restriction $f|_\Lambda$ admits a dense orbit;

(vii) periodic points of f of arbitrarily large prime periods are dense in Λ;

(viii) the restriction $f|_\Lambda$ is chaotic.

Idea of the proof: Suppose that for some $\mathbf{s} \in \Sigma_2$ the preimage $S^{-1}(\mathbf{s})$ is a nontrivial segment $[a,b]$. If $\mathbf{s}$ is not periodic then (a,b) is a wandering domain. Otherwise some iterate of f is a monotone transformation of $[a,b]$.

Expansiveness

Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map, Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0,1]$.

Proposition 1 Suppose f is continuously differentiable and $|f'(x)| > 1$ for all $x \in \Lambda$. Then the restriction $f|_\Lambda$ is expansive.

Proposition 2 The restriction $f|_\Lambda$ is expansive if and only if some iterate $f^n|_\Lambda$ is expansive.

Theorem Suppose f is continuously differentiable and $|(f^n)'(x)| > 1$ for some $n \geq 1$ and all $x \in \Lambda$. Then the restriction $f|_\Lambda$ is expansive.

Remark. The logistic map $f(x) = \mu x(1-x)$ satisfies the hypothesis of Proposition 1 for $\mu > 2 + \sqrt{5}$. It satisfies the hypothesis of the theorem for all $\mu > 4$.

Structural stability

Informally, a dynamical system is **structurally stable** if its structure is preserved under small perturbations. To make this notion formal, one has to specify what it means that the "structure is preserved" and what is considered a "small perturbation".

In the context of topological dynamics, structural stability usually means that the perturbed system is topologically conjugate to the original one.

The description of small perturbations varies for different dynamical systems and so there are various kinds of structural stability.

- Structural stability within a parametric family.

Suppose $f_{\mathbf{p}} : X_{\mathbf{p}} \to X_{\mathbf{p}}$ is a dynamical system depending on a parameter vector $\mathbf{p} \in P$, where $P \subset \mathbb{R}^k$. Given $\mathbf{p}_0 \in P$, we say that $f_{\mathbf{p}_0}$ is **structurally stable within the family** $\{f_{\mathbf{p}}\}$ if there exists $\varepsilon > 0$ such that for any $\mathbf{p} \in P$ satisfying $|\mathbf{p} - \mathbf{p}_0| < \varepsilon$ the system $f_{\mathbf{p}}$ is topologically conjugate to $f_{\mathbf{p}_0}$.

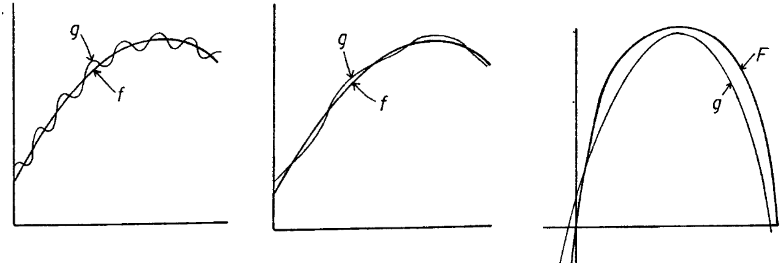
- C^r-structural stability for one-dimensional systems.

Let J be an interval of the real line. For any integer $r \geq 0$, let $C^r(J)$ denote the set of r times continuously differentiable functions $f : J \to \mathbb{R}$. The C^r distance between functions $f, g \in C^r(J)$ is given by

$$d_r(f,g) = \sup_{x \in J} \big(|f(x)-g(x)|, |f'(x)-g'(x)|, \ldots, |f^{(r)}(x)-g^{(r)}(x)|\big).$$

We say that a map $f \in C^r(J)$ is C^r-**structurally stable** if there exists $\varepsilon > 0$ such that whenever $d_r(f,g) < \varepsilon$, it follows that g is topologically conjugate to f.

Small perturbation



In the first figure, the function g is a C^0-small perturbation of f, but not a C^1-small one. In the second figure, the functions f and g are C^1-close but not C^2-close. In the third figure, f and g are C^2-close.

Examples of structural stability

- Logistic map $F_\mu : \mathbb{R} \to \mathbb{R}$, $F_\mu(x) = \mu x(1-x)$.

The map F_μ is structurally stable within the family $\{F_\mu\}$ for $\mu > 4$. Besides, it is C^2-structurally stable for $\mu > 4$ (but not C^1-structurally stable). It is C^1-structurally stable for $\mu > 4$ within the family of unimodal maps.