MATH 614

Dynamical Systems and Chaos

Lecture 11: Structural stability (continued). Sharkovskii's theorem.

- Structural stability within a parametric family.

Suppose $f_{\mathbf{p}} : X_{\mathbf{p}} \to X_{\mathbf{p}}$ is a dynamical system depending on a parameter vector $\mathbf{p} \in P$, where $P \subset \mathbb{R}^k$. Given $\mathbf{p}_0 \in P$, we say that $f_{\mathbf{p}_0}$ is **structurally stable within the family** $\{f_{\mathbf{p}}\}$ if there exists $\varepsilon > 0$ such that for any $\mathbf{p} \in P$ satisfying $|\mathbf{p} - \mathbf{p}_0| < \varepsilon$ the system $f_{\mathbf{p}}$ is topologically conjugate to $f_{\mathbf{p}_0}$.

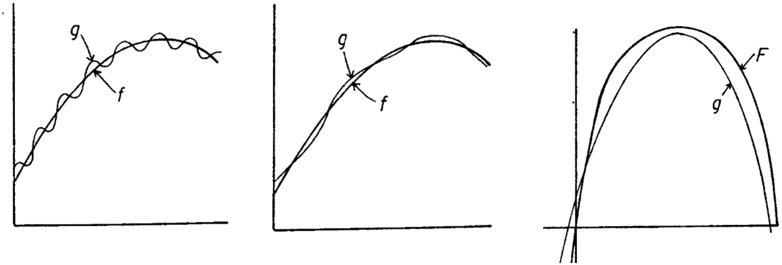
- C^r-structural stability for one-dimensional systems.

Let J be an interval of the real line. For any integer $r \geq 0$, let $C^r(J)$ denote the set of r times continuously differentiable functions $f : J \to \mathbb{R}$. The C^r distance between functions $f, g \in C^r(J)$ is given by

$$d_r(f,g) = \sup_{x \in J}\big(|f(x) - g(x)|, |f'(x) - g'(x)|, \ldots, |f^{(r)}(x) - g^{(r)}(x)|\big).$$

We say that a map $f \in C^r(J)$ is C^r-**structurally stable** if there exists $\varepsilon > 0$ such that whenever $d_r(f,g) < \varepsilon$, it follows that g is topologically conjugate to f.

Small perturbation

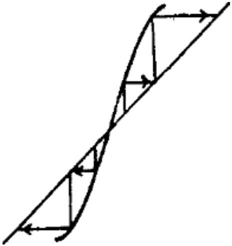


In the first figure, the function g is a C^0-small perturbation of f, but not a C^1-small one. In the second figure, the functions f and g are C^1-close but not C^2-close. In the third figure, f and g are C^2-close.

Examples of structural stability

- Linear map $f_\lambda : \mathbb{R} \to \mathbb{R}$, $f_\lambda(x) = \lambda x$.

The map f_λ is structurally stable within the family $\{f_\lambda\}$ if and only if $\lambda \notin \{-1, 0, 1\}$. Besides, it is C^1-structurally stable for the same values of λ.



Proposition Suppose $f : [0, \infty) \to [0, \infty)$ and $g : [0, \infty) \to [0, \infty)$ are invertible continuous maps such that $f(0) = g(0) = 0$, $f(x) > x$ and $g(x) > x$ for all $x > 0$. Then f and g are topologically conjugate.

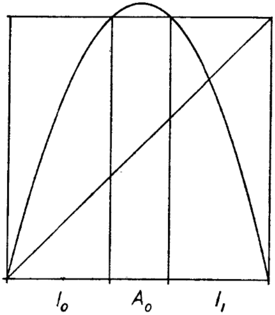
Let $a, b > 0$. We can construct a continuous conjugacy ϕ such that $\phi(f^n(a)) = g^n(b)$ for all n. Let $U = (a, f(a))$. Then U is a **wandering domain** of the map f, which means that sets $\ldots, f^{-1}(U), U, f(U), f^2(U), \ldots$ are disjoint. Similarly, $V = (b, g(b))$ is a wandering domain of g.

$$
\begin{array}{ccccccccccc}
\xrightarrow{f} & f^{-1}(U) & \xrightarrow{f} & U & \xrightarrow{f} & f(U) & \xrightarrow{f} & f^2(U) & \xrightarrow{f} \\
 & & & \phi \big\downarrow & & & & & \\
\xrightarrow{g} & g^{-1}(V) & \xrightarrow{g} & V & \xrightarrow{g} & g(V) & \xrightarrow{g} & g^2(V) & \xrightarrow{g}
\end{array}
$$

Examples of structural stability

- Logistic map $F_\mu : \mathbb{R} \to \mathbb{R}$, $F_\mu(x) = \mu x(1-x)$.

The map F_μ is structurally stable within the family $\{F_\mu\}$ for $\mu > 4$. Besides, it is C^2-structurally stable for $\mu > 4$ (but not C^1-structurally stable). It is C^1-structurally stable for $\mu > 4$ within the family of unimodal maps.



Period set

Suppose J is an interval of the real line and $f : J \to J$ is a continuous map. Let $\mathcal{P}(f)$ be the set of all natural numbers n for which the map f admits a periodic point of prime period n (or, equivalently, a periodic orbit that consists of n points).

Question. Which subsets of $\mathbb{N}$ can occur as $\mathcal{P}(f)$?

Examples. • $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$.
$\mathcal{P}(f) = \emptyset$.

• $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$.
$\mathcal{P}(f) = \{1\}$.

• $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x$.
$\mathcal{P}(f) = \{1, 2\}$.

• $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \mu x(1 - x)$, where $\mu > 4$.
The map f has an invariant set Λ such that the restriction $f|_\Lambda$ is conjugate to the shift on $\Sigma_{\{0,1\}}$. Since the shift admits periodic points of all prime periods, so does f: $\mathcal{P}(f) = \mathbb{N}$.

Sharkovskii's ordering

The **Sharkovskii ordering** is the following strict linear ordering of the natural numbers:

$$\begin{array}{cccccccccc} & 3 & \rhd & 5 & \rhd & 7 & \rhd & 9 & \rhd & \ldots \\ \rhd & 2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & 2\cdot 9 & \rhd & \ldots \\ \rhd & 2^2\cdot 3 & \rhd & 2^2\cdot 5 & \rhd & 2^2\cdot 7 & \rhd & 2^2\cdot 9 & \rhd & \ldots \end{array}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots \rhd 2^k \rhd \ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

To be precise, for any integers $k_1, k_2 \geq 0$ and odd natural numbers p_1, p_2 we let $2^{k_1}p_1 \rhd 2^{k_2}p_2$ if and only if one of the following conditions holds:

- $k_1 = k_2$ and $1 < p_1 < p_2$;
- $p_1, p_2 > 1$ and $k_1 < k_2$;
- $p_1 > 1$ and $p_2 = 1$;
- $p_1 = p_2 = 1$ and $k_1 > k_2$.

Sharkovskii's Theorem

Theorem 1 (Sharkovskii) Suppose $f : J \to J$ is a continuous map of an interval $J \subset \mathbb{R}$. If f admits a periodic point of prime period n and $n \triangleright m$ for some $m \in \mathbb{N}$, then f admits a periodic point of prime period m as well.

Definition. A subset $E \subset \mathbb{N}$ is called a **tail** of Sharkovskii's ordering if $n \in E$ and $n \triangleright m$ implies $m \in E$ for all $m, n \in \mathbb{N}$.

Sharkovskii's Theorem states that the period set $\mathcal{P}(f)$ is such a tail. For any $n \in \mathbb{N}$ the set $E_n = \{n\} \cup \{m \in \mathbb{N} \mid n \triangleright m\}$ is a tail. The only tails that cannot be represented this way are $\{2^n \mid n \geq 0\}$ and the empty set.

Theorem 2 For any tail E of Sharkovskii's ordering there exists a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{P}(f) = E$.

Remark. For maps of an interval $J \subset \mathbb{R}$, Theorem 2 holds with one exception: if J is bounded and closed, then $\mathcal{P}(f) \neq \emptyset$.

Suppose $f : J \to J$ is a continuous map of an interval $J \subset \mathbb{R}$. Given two closed bounded intervals $I_1, I_2 \subset J$, we write $\boxed{I_1 \to I_2}$ if $f(I_1) \supset I_2$ (i.e., if I_1 covers I_2 under the action of f).

Lemma 1 If $I \to I$, then the interval I contains a fixed point of the map f.

Proof: Let $I = [a, b]$. Since $f(I) \supset I$, there exist $a_0, b_0 \in I$ such that $f(a_0) = a$, $f(b_0) = b$. Then a continuous function $g(x) = f(x) - x$ satisfies $g(a_0) = a - a_0 \leq 0$ and $g(b_0) = b - b_0 \geq 0$. By the Intermediate Value Theorem, we have $g(c) = 0$ for some c between a_0 and b_0. Then $c \in I$ and $f(c) = c$.