Dynamical Systems and Chaos

MATH 614

Lecture 16: Rotation number.

The standard family.

Maps of the circle

 $T:S^1\to S^1$,

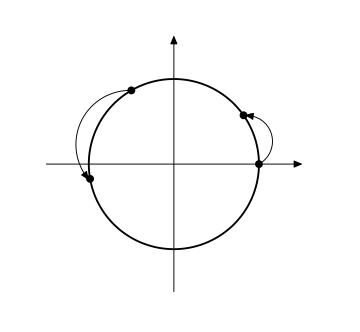
 ${\cal T}$ an orientation-preserving homeomorphism.

Suppose $T:S^1\to S^1$ is an orientation-preserving homeomorphism.

Is T topologically conjugate to a rotation R_{ω} ? Assume this is so, then how can we find ω ?

For any $x \in S^1$ let $\omega(T, x)$ denote the length of the shortest arc that goes from x to T(x) in the counterclockwise direction.

If T is a rotation then $\omega(T,x)$ is a constant.



Rotation number

Consider the average

$$A_n(T,x) = \frac{1}{n} \sum_{k=0}^{n-1} \omega(T, T^k(x)).$$

Theorem The limit

$$\lim_{n\to\infty}A_n(T,x)$$

exists for any $x \in S^1$ and does not depend on x.

The **rotation number** of T is

$$\rho(T) = \frac{1}{2\pi} \lim_{n \to \infty} A_n(T, x).$$

Proposition If $\rho(T) = 0$, then T has a fixed point.

Proof: Suppose T has no fixed points. Then $0 < \omega(T,x) < 2\pi$ for any $x \in S^1$. Since $\omega(T,x)$ is a continuous function of x, there exists $\varepsilon > 0$ such that $\varepsilon \leq \omega(T,x) \leq 2\pi - \varepsilon$ for any $x \in S^1$. Then $\varepsilon \leq A_n(T,x) \leq 2\pi - \varepsilon$ for all $x \in S^1$ and $n = 1, 2, \ldots$ It follows that

$$\frac{\varepsilon}{2\pi} \le \rho(T) \le 1 - \frac{\varepsilon}{2\pi}$$
.

Properties of the rotation number

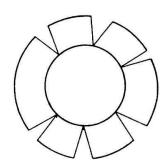
- For any T, $0 \le \rho(T) < 1$.
- $\rho(R_{\omega}) = \omega/(2\pi) \pmod{1}$, where R_{ω} is the rotation by ω .
- If g is an orientation-preserving homeomorphism of S^1 , then $\rho(g^{-1}Tg) = \rho(T)$.
- If g is an orientation-reversing homeomorphism of S^1 , then $\rho(g^{-1}Tg) = -\rho(T) \pmod{1}$.
- If T_1 and T_2 are topologically conjugate, then $\rho(T_1) = \pm \rho(T_2) \pmod{1}$.

Properties of the rotation number

- Rotations R_{ω_1} and R_{ω_1} are topologically conjugate if and only if $\omega_1 = \pm \omega_2 \pmod{2\pi}$.
 - $\rho(T^n) = n\rho(T) \pmod{1}$.
 - $\rho(T) = 0$ if and only if T has a fixed point.
- $\rho(T)$ is rational if and only if T has a periodic point.
- If T has a periodic point of prime period n, then $\rho(T) = k/n$, a reduced fraction.

Theorem (Denjoy) If T is C^2 smooth and the rotation number $\rho(T)$ is irrational, then T is topologically conjugate to a rotation of the circle.

Example (Denjoy). There exists C^1 smooth diffeomorphism T of S^1 such that $\rho(T)$ is irrational but T is not minimal.



Proposition Suppose $f: S^1 \to S^1$ is an orientation-preserving homeomorphism. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for any homeomorphism $g: S^1 \to S^1$ with

$$\sup_{x \in S^1} \operatorname{dist}(f(x), g(x)) < \delta$$

we have $|\rho(f) - \rho(g)| < \varepsilon \pmod{1}$.

Corollary Suppose f_{λ} is a one-parameter family of orientation-preserving homeomorphisms of S^1 . If f_{λ} depends continuously on λ then $\rho(f_{\lambda})$ is a continuous $\pmod{1}$ function of λ .

The standard family

The **standard** (or **canonical**) family of maps

$$f_{\omega,\varepsilon}:S^1 o S^1,\quad \omega\in\mathbb{R},\ arepsilon\geq 0.$$

In the angular coordinate α :

$$f_{\omega,\varepsilon}(\alpha) = \alpha + \omega + \varepsilon \sin \alpha.$$

If $\varepsilon=0$ then $f_{\omega,\varepsilon}=R_\omega$ is a rotation. For $0\leq \varepsilon<1$, $f_{\omega,\varepsilon}$ is a diffeomorphism. If $\varepsilon=1$ then $f_{\omega,\varepsilon}$ is only a homeomorphism. If $\varepsilon>1$ then $f_{\omega,\varepsilon}$ is not one-to-one.

The rotation number $\rho(f_{\omega,\varepsilon})$:

- depends continuously (mod 1) on ω and ε ;
- is a 2π -periodic function of ω for any ε ;
- $f_{0,\varepsilon}$ has rotation number 0;
- $\rho(f_{\omega,\varepsilon})$ is a non-decreasing function of $\omega \in (0,2\pi)$ for any fixed ε ;
 - $\lim_{\omega \to 2\pi} \rho(f_{\omega,\varepsilon}) = 1.$

Hence the map $r_{\varepsilon}:[0,1)\to[0,1)$ given by $x\mapsto \rho(f_{2\pi \times,\varepsilon})$ is continuous, non-decreasing, and onto.

 r_0 is the identity.

Proposition Suppose $\rho(f_{\omega_0,\varepsilon})$ is rational. If $\varepsilon>0$ then

$$\rho(f_{\omega,\varepsilon}) = \rho(f_{\omega_0,\varepsilon})$$

for all $\omega > \omega_0$ close enough to ω_0 or for all $\omega < \omega_0$ close enough to ω_0 (or both).

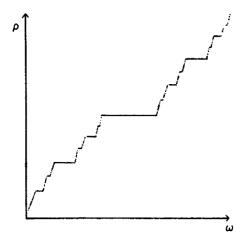
Theorem For any irrational number $0 < \rho_0 < 1$ and any $0 < \varepsilon < 1$, there is exactly one $\omega \in (0, 2\pi)$ such that $\rho(f_{\omega,\varepsilon}) = \rho_0$.

Let $0 < \varepsilon < 1$ and $0 \le \rho_0 < 1$. Then $r_{\varepsilon}^{-1}(\rho_0)$ is a point if ρ_0 is irrational and $r_{\varepsilon}^{-1}(\rho_0)$ is a nontrivial interval if ρ_0 is rational.

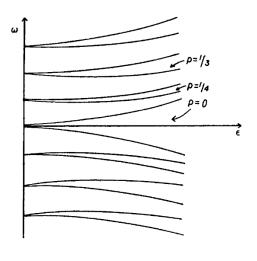
 r_{ε} is a **Cantor function**, which means that on the complement of a Cantor set, $r'_{\varepsilon} = 0$.

The graph of a Cantor function is called the "devil's staircase".

Cantor function



The bifurcation diagram for the standard family



The bifurcation diagram for the standard family

We plot the regions in the (ε,ω) -plane where $\rho(f_{\omega,\varepsilon})$ is a fixed rational number. Each region is a "tongue" that flares from a point $\varepsilon=0,\ \omega=m/n,$ $m,n\in\mathbb{Z}$. None of these tongues can overlap when $\varepsilon<1$.

Consider the tongue corresponding to $\rho=0$. It describes fixed points of the standard maps. This tongue is the angle $|\omega|\leq \varepsilon$.

What happens when we fix ε and vary ω ?

If $\omega = -\varepsilon$ then $f_{\omega,\varepsilon}(\alpha) = \alpha + \omega + \varepsilon \sin \alpha$ has a unique fixed point $\pi/2$. As we increase ω , it splits into two fixed points, one in $(-\pi/2, \pi/2)$, the other in $(\pi/2, 3\pi/2)$. They run around the circle in opposite directions. Finally, at $\omega = \varepsilon$ the two points coalesce into a single fixed point $-\pi/2$. The unique fixed points for $\omega = \pm \varepsilon$ are neutral. As for two fixed points for $|\omega| < \varepsilon$, one is

attracting while the other is repelling (which one?). So the family $f_{\omega,\varepsilon}$ (ε fixed) enjoys a saddle-node bifurcation two times. Notice that these are not pure saddle-node bifurcations since the bifurcation points are not isolated (they are "half-isolated").