MATH 614

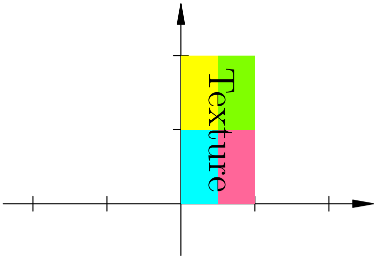
Dynamical Systems and Chaos

Lecture 17b: Dynamics of linear maps.

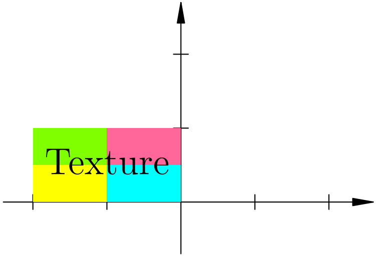
Linear transformations

Any linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is represented as multiplication of an n-dimensional column vector by a $n \times n$ matrix: $L(\mathbf{x}) = A\mathbf{x}$, where $A = (a_{ij})_{1 \leq i,j \leq n}$.

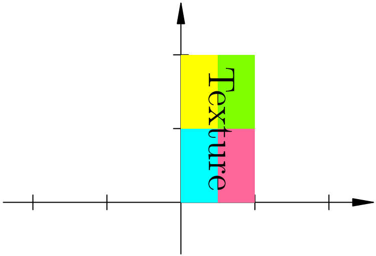
Dynamics of linear transformations corresponding to particular matrices is determined by eigenvalues and the Jordan canonical form.



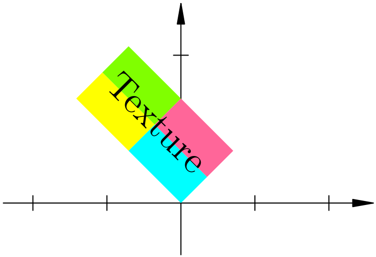
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



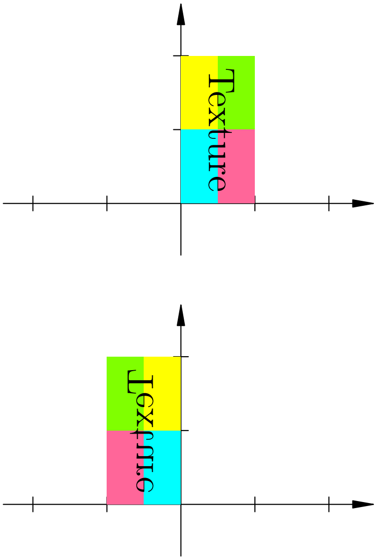
Rotation by 90°



$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

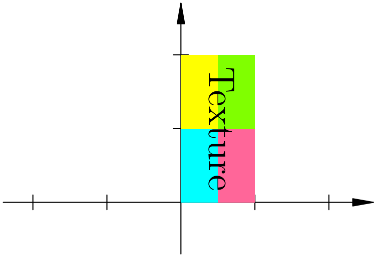


Rotation by 45^{o}

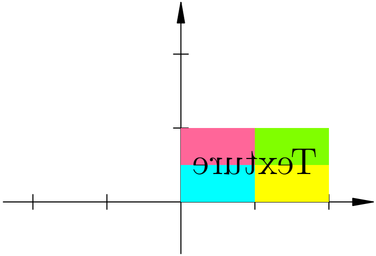


$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

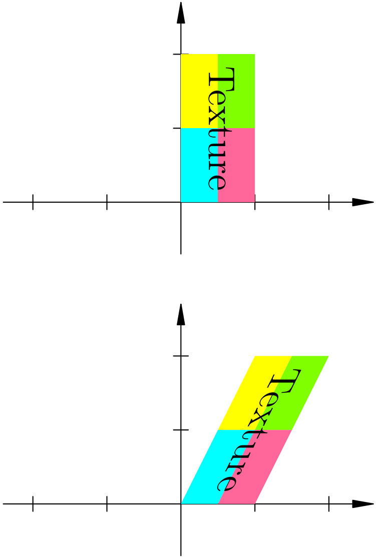
Reflection about
the vertical axis



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

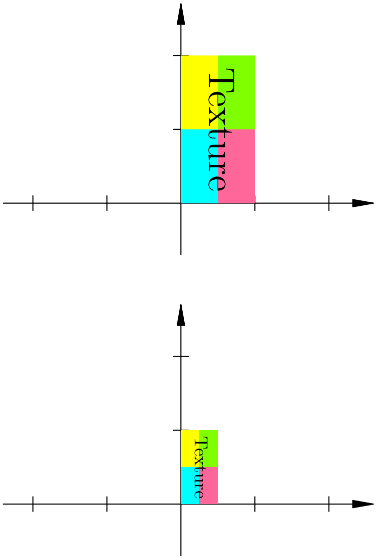


Reflection about the line $x - y = 0$



$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

Horizontal shear

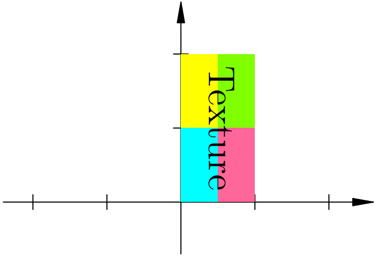


$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

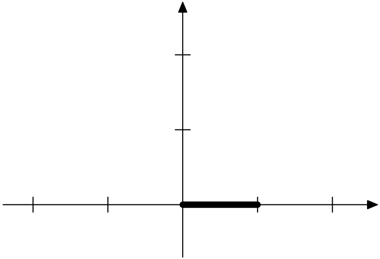
Scaling

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$$

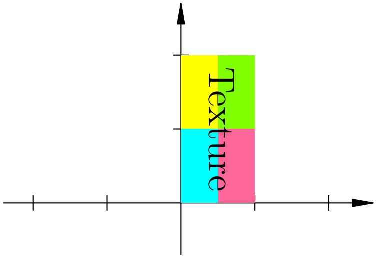
Squeeze



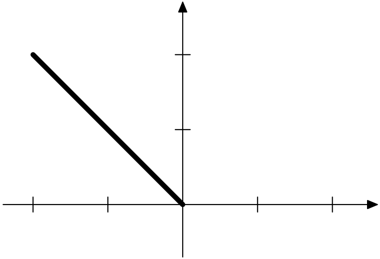
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



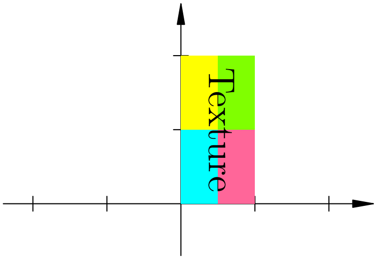
Vertical projection on the horizontal axis



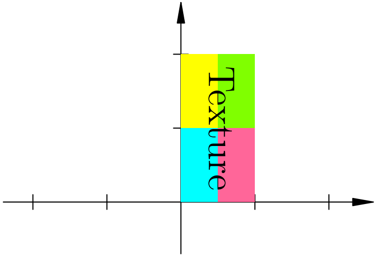
$$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$



Horizontal projection
on the line $x + y = 0$



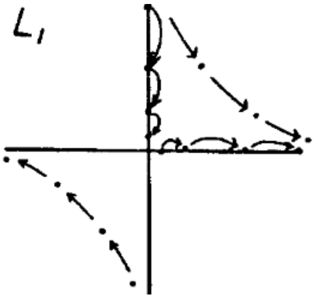
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

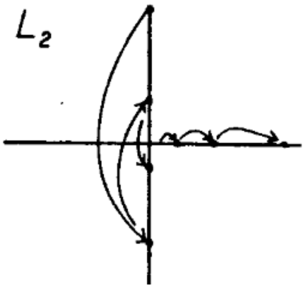


Identity

Phase portraits of linear maps

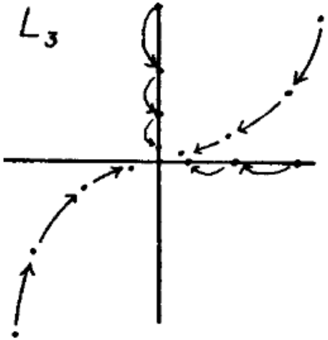
$$L_1(\mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \mathbf{x} \qquad L_2(\mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & -1/2 \end{pmatrix} \mathbf{x}$$

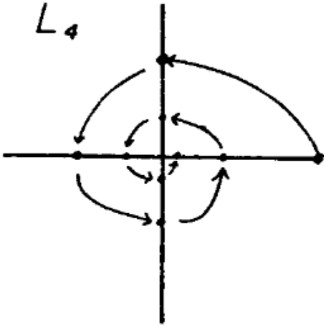




Phase portraits of linear maps

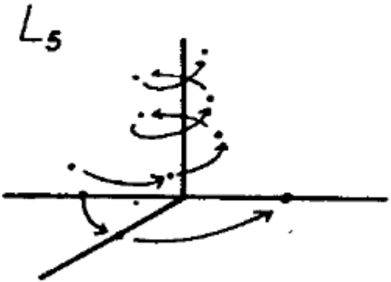
$$L_3(\mathbf{x}) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \mathbf{x} \qquad L_4(\mathbf{x}) = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \mathbf{x}$$





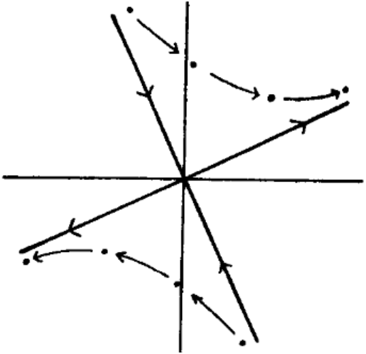
Phase portraits of linear maps

$$L_5(\mathbf{x}) = \begin{pmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$$



Phase portraits of linear maps

$$L(\mathbf{x}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x}$$



Stable and unstable subspaces

Proposition 1 Suppose that all eigenvalues of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ are less than 1 in absolute value. Then $L^n(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proposition 2 Suppose that all eigenvalues of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ are greater than 1 in absolute value. Then $L^{-n}(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$ for all $\mathbf{x} \in \mathbb{R}^n$.

Given a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$, let W^s denote the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $L^n(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$. In the case L is invertible, let W^u denote the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $L^{-n}(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$.

Proposition 3 W^s and W^u are vector subspaces of $\mathbb{R}^n$ that are transversal: $W^s \cap W^u = \{\mathbf{0}\}$.

Definition. W^s is called the **stable subspace** of the linear map L. W^u is called the **unstable subspace** of L.