MATH 614

Dynamical Systems and Chaos

Lecture 21:
Markov partitions.
Solenoid.

General symbolic dynamics

Suppose $f : X \to X$ is a dynamical system. Given a partition of the set X into disjoint subsets X_α, $\alpha \in \mathcal{A}$ indexed by elements of a finite set $\mathcal{A}$, we can define the (forward) **itinerary map** $S : X \to \Sigma_{\mathcal{A}}$ so that $S(x) = (s_0 s_1 s_2 \dots)$, where $f^n(x) \in X_{s_n}$ for all $n \geq 0$.

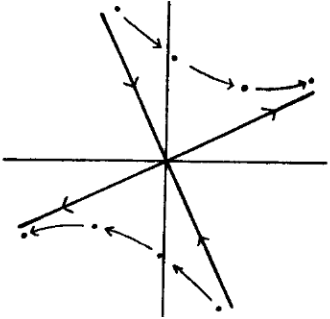
If the map f is invertible, then we can define the full itinerary map $S : X \to \Sigma_{\mathcal{A}}^{\pm}$.

In the case f is continuous, the itinerary map is continuous if the sets X_α are **clopen** (i.e., both closed and open). If, additionally, X is compact, then the itinerary map provides a semi-conjugacy of f with a subshift.

In the case a partition into clopen sets is not possible, we can choose closed sets X_α that do not cover X completely or closed sets that partially overlap.

Examples of stable and unstable sets

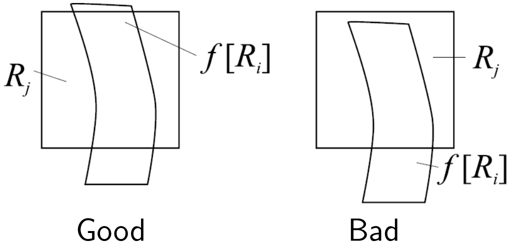
- Hyperbolic toral automorphism $L_A : \mathbb{T}^2 \to \mathbb{T}^2$.



Stable and unstable sets of L_A are images of the corresponding sets of the linear map $L(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^2$, under the natural projection $\pi : \mathbb{R}^2 \to \mathbb{T}^2$. These sets are dense in the torus $\mathbb{T}^2$.

Markov partitions

Definition. Given a metric space M and a homeomorphism $f : M \to M$, a **rectangle** is a closed set $R \subset M$ such that for any $\mathbf{p}, \mathbf{q} \in R$, the intersection $W^s(\mathbf{p}) \cap W^u(\mathbf{q}) \cap R$ is not empty. A **Markov partition** of M is a partition of M into rectangles $\{R_1, \ldots, R_m\}$ with disjoint interiors such that whenever $\mathbf{p} \in R_i$ and $f(\mathbf{p}) \in R_j$, we have $f\big(W^u(\mathbf{p}) \cap R_i\big) \supset W^u(f(\mathbf{p})) \cap R_j$ and $f\big(W^s(\mathbf{p}) \cap R_i\big) \subset W^s(f(\mathbf{p})) \cap R_j$.



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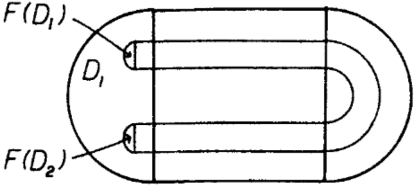
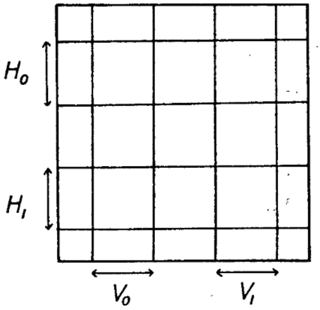
The conditions ensure that $f^n(R_i) \cap R_j \neq \emptyset$ and $f^m(R_j) \cap R_k \neq \emptyset$ implies $f^{n+m}(R_i) \cap R_k \neq \emptyset$ so that the corresponding symbolic dynamics is a topological Markov chain.

Note that all points in $W^s(\mathbf{p}) \cap R_i$ have the same forward itinerary while all points in $W^u(\mathbf{p}) \cap R_i$ have the same backward itinerary.

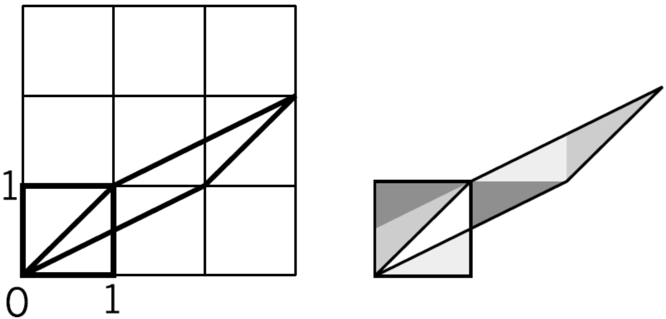
Example





Cat map

The **cat map** is a hyperbolic toral automorphism $L_A : \mathbb{T}^2 \to \mathbb{T}^2$ given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.



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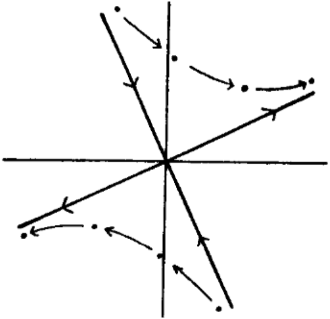

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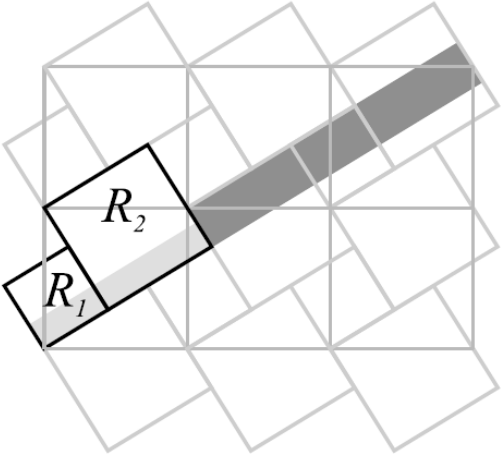
Markov partition for the cat map



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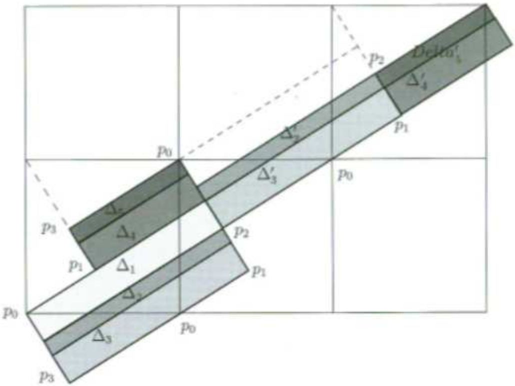


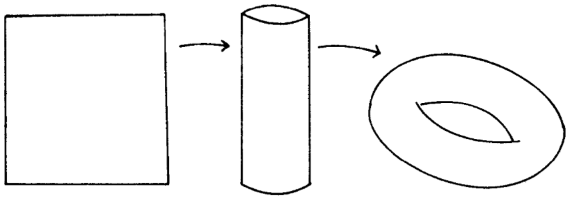
Adler, Weiss 1967

Solid torus

Let S^1 be the circle and B^2 be the unit disk in $\mathbb{R}^2$:

$$B^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

The Cartesian product $D = S^1 \times B^2$ is called the **solid torus**. It is a 3-dimensional manifold with boundary that can be realized as a closed subset in $\mathbb{R}^3$. The boundary ∂D is the torus.



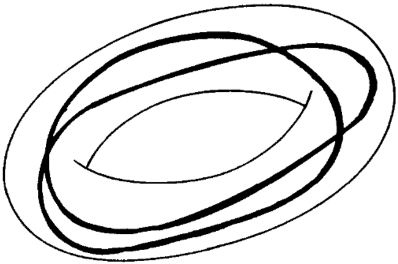
Let $D = S^1 \times B^2$ be the solid torus. We represent the circle S^1 as $\mathbb{R}/\mathbb{Z}$. For any $\theta \in S^1$ and $p \in B^2$ let

$$F(\theta, p) = \big(2\theta, ap + b\phi(\theta)\big),$$

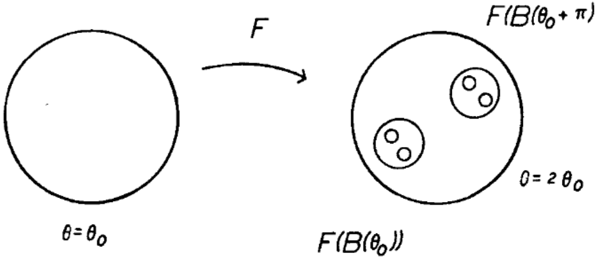
where $\phi : S^1 \to \partial B^2$ is defined by

$$\phi(\theta) = \big(\cos(2\pi\theta), \sin(2\pi\theta)\big)$$

and constants a, b are chosen so that $0 < a < b$ and $a + b < 1$. Then $F : D \to D$ is a smooth, one-to-one map. The image $F(D)$ is contained strictly inside of D.



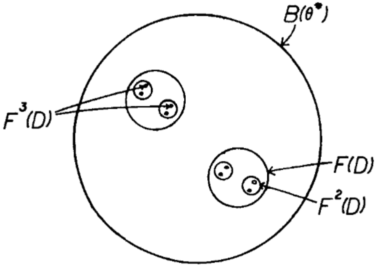
The solid torus $D = S^1 \times B^2$ is foliated by discs $B(\theta) = \{\theta\} \times B^2$. The image $F(B(\theta))$ is a smaller disc inside of $B(2\theta)$.



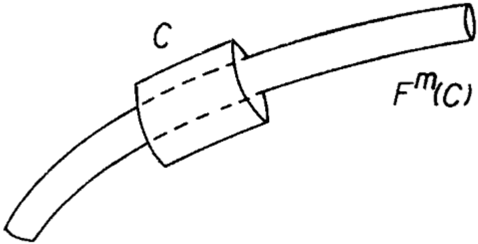
It follows that all points in a disc $B(\theta)$ are forward asymptotic. In particular, $B(\theta)$ is contained in the stable set $W^s(\mathbf{x})$ of any point $\mathbf{x} \in B(\theta)$. In fact, $W^s(\mathbf{x}) = \bigcup_{n,k \in \mathbb{Z}} B(\theta + n/2^k)$.

Solenoid

The sets $D, F(D), F^2(D), \ldots$ are closed and nested. The intersection $\Lambda = \bigcap_{n \geq 0} F^n(D)$ is called the **solenoid**.



The solenoid Λ is a compact set invariant under the map F. The restriction of F to Λ is an invertible map. The intersection of Λ with any disc $B(\theta)$ is a Cantor set. Moreover, Λ is locally the Cartesian product of a Cantor set and an arc.


C
$F^m(C)$

Properties of the solenoid

Theorem 1 The restriction $F|_\Lambda$ is chaotic, i.e.,

- it has sensitive dependence on initial conditions,
- periodic points are dense in Λ,
- it is topologically transitive.

Theorem 2 The solenoid Λ is an attractor of the map F. Namely, $\mathrm{dist}\big(F^n(\mathbf{x}), \Lambda\big) \to 0$ as $n \to \infty$ for all $\mathbf{x} \in D$.

Theorem 3 For any point $\mathbf{x} \in \Lambda$, the unstable set $W^u(\mathbf{x})$ is a smooth curve that is dense in Λ.

Theorem 4 The solenoid is connected, but not locally connected or arcwise connected.