MATH 614

Dynamical Systems and Chaos

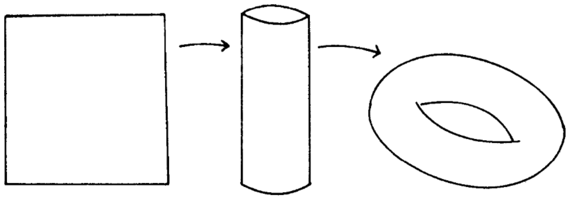
Lecture 22: Solenoid (continued). Attractors.

Solid torus

Let S^1 be the circle and B^2 be the unit disk in $\mathbb{R}^2$:

$$B^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

The Cartesian product $D = S^1 \times B^2$ is called the **solid torus**. It is a 3-dimensional manifold with boundary that can be realized as a closed subset in $\mathbb{R}^3$. The boundary ∂D is the torus.



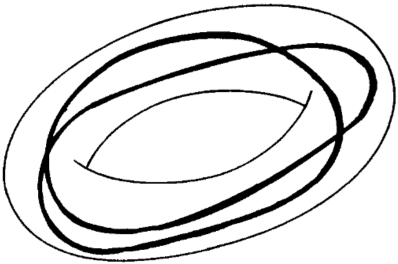
Let $D = S^1 \times B^2$ be the solid torus. We represent the circle S^1 as $\mathbb{R}/\mathbb{Z}$. For any $\theta \in S^1$ and $p \in B^2$ let

$$F(\theta, p) = \big(2\theta, ap + b\phi(\theta)\big),$$

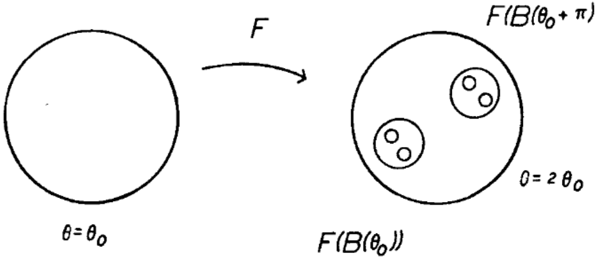
where $\phi : S^1 \to \partial B^2$ is defined by

$$\phi(\theta) = \big(\cos(2\pi\theta), \sin(2\pi\theta)\big)$$

and constants a, b are chosen so that $0 < a < b$ and $a + b < 1$. Then $F : D \to D$ is a smooth, one-to-one map. The image $F(D)$ is contained strictly inside of D.



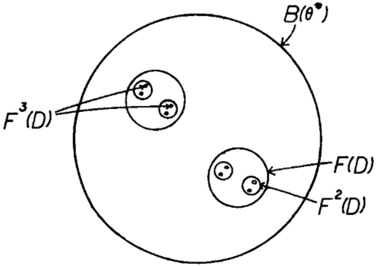
The solid torus $D = S^1 \times B^2$ is foliated by discs $B(\theta) = \{\theta\} \times B^2$. The image $F(B(\theta))$ is a smaller disc inside of $B(2\theta)$.



It follows that all points in a disc $B(\theta)$ are forward asymptotic. In particular, $B(\theta)$ is contained in the stable set $W^s(\mathbf{x})$ of any point $\mathbf{x} \in B(\theta)$. In fact, $W^s(\mathbf{x}) = \bigcup_{n,k\in\mathbb{Z}} B(\theta + n/2^k)$.

Solenoid

The sets $D, F(D), F^2(D), \ldots$ are closed and nested. The intersection $\Lambda = \bigcap_{n \geq 0} F^n(D)$ is called the **solenoid**.



The solenoid Λ is a compact set invariant under the map F. The restriction of F to Λ is an invertible map. The intersection of Λ with any disc $B(\theta)$ is a Cantor set. Moreover, Λ is locally the Cartesian product of a Cantor set and an arc.

Properties of the solenoid

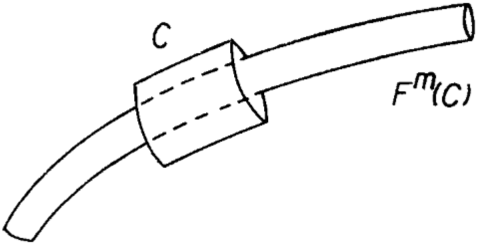
Theorem 1 The restriction $F|_\Lambda$ is chaotic, i.e.,

- it has sensitive dependence on initial conditions,
- it is topologically transitive,
- periodic points are dense in Λ.

Theorem 2 The solenoid Λ is an attractor of the map F. Namely, $\mathrm{dist}\big(F^n(\mathbf{x}), \Lambda\big) \to 0$ as $n \to \infty$ for all $\mathbf{x} \in D$.

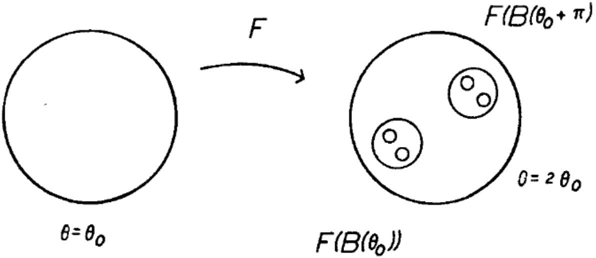
Theorem 3 For any point $\mathbf{x} \in \Lambda$, the unstable set $W^u(\mathbf{x})$ is a smooth curve that is dense in Λ.

Theorem 4 The solenoid is connected, but not locally connected or arcwise connected.


C
$F^m(C)$

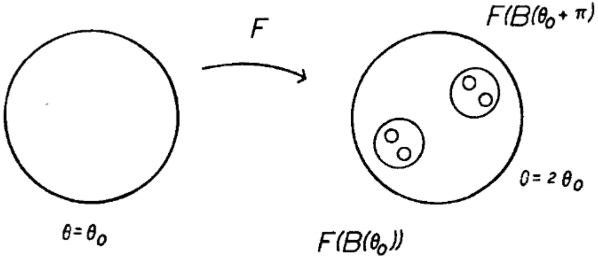
Periodic points

The solid torus $D = S^1 \times B^2$ is foliated by discs $B(\theta) = \{\theta\} \times B^2$. The image $F(B(\theta))$ is a smaller disc inside of $B(2\theta)$.



If θ is a periodic point of the doubling map, then $B(\theta)$ contains a unique periodic point of F (of the same period).

Symbolic dynamics



Let $(\theta, p) \in \Lambda$ and consider the full orbit

$$\ldots, (\theta_{-2}, p_{-2}), (\theta_{-1}, p_{-1}), (\theta_0, p_0), (\theta_1, p_1), (\theta_2, p_2), \ldots,$$

where $(\theta_n, p_n) = F^n(\theta, p)$. It turns out that (θ, p) can be uniquely recovered from the sequence

$$\ldots, \theta_{-2}, \theta_{-1}, \theta_0, \theta_1, \theta_2, \ldots$$

Even more, it is enough to consider the itinerary relative to the partition $S^1 = [0, 1/2] \cup [1/2, 1]$.

Inverse limit space extension

Suppose $f : X \to X$ is a dynamical system (X a compact metric space, f a continuous map) that is not invertible. We can associate an invertible dynamical system to it as follows.

Since $f(X) \subset X$, it follows that $X \supset f(X) \supset f^2(X) \supset \dots$ Hence $X, f(X), f^2(X), \dots$ are nested compact sets so that $Y = X \cap f(X) \cap f^2(X) \cap \dots$ is a nonempty compact set. It is invariant under f and the restriction $f|_Y$ is onto.

Since f maps Y onto itself, we can think of f^{-1} as a multi-valued function on Y. Let Z denote the set of all possible backward orbits of f, i.e., sequences $(x_0, x_1, x_2, \dots)$ such that $\dots \overset{f}{\mapsto} x_2 \overset{f}{\mapsto} x_1 \overset{f}{\mapsto} x_0$. The shift map is well defined on Z and it is invertible. Let F denote the inverse. Then the map $\phi : Z \to Y$ given by $\phi(x_0, x_1, x_2, \dots) = x_0$ is a semi-conjugacy of F with $f|_Y$. The infinite product $Y \times Y \times \dots$ is naturally endowed with a topology so that the set $Z \subset Y^\infty$ is compact while maps F and ϕ are continuous.

Examples

- One-sided shift $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$, $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 \dots)$.

The inverse limit space extension of σ is topologically conjugate to the two-sided shift $\sigma : \Sigma^{\pm}_{\mathcal{A}} \to \Sigma^{\pm}_{\mathcal{A}}$ over the same alphabet.

- Doubling map $D : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $D(\theta) = 2\theta \,(\mathrm{mod}\ 1)$.

The inverse limit space extension of D is topologically conjugate to the solenoid map.

Attractors

Suppose $F : D \to D$ is a topological dynamical system on a metric space D.

Definition. A compact set $N \subset D$ is called a **trapping region** for F if $F(N) \subset \mathrm{int}(N)$.

If N is a trapping region, then $N, F(N), F^2(N), \dots$ are nested compact sets and their intersection Λ is an invariant set: $F(\Lambda) \subset \Lambda$.

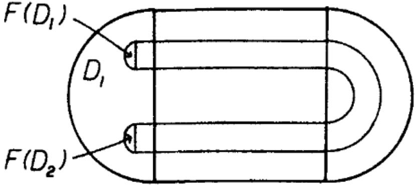
Definition. A set $\Lambda \subset D$ is called an **attractor** for F if there exists a neighborhood N of Λ such that the closure $\overline{N}$ is a trapping region for F and $\Lambda = N \cap F(N) \cap F^2(N) \cap \dots$

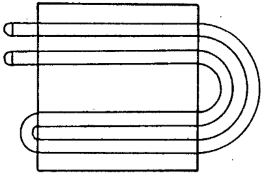
The attractor Λ is **transitive** if the restriction of F to Λ is a transitive map.

Examples of attractors

- Any attracting fixed point or an attracting periodic orbit is a transitive attractor.

- The solenoid is a transitive attractor.





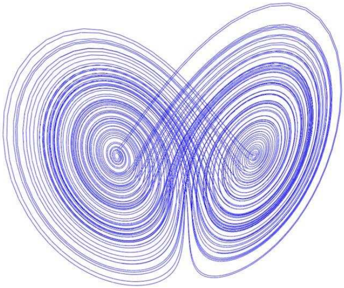
- The horseshoe map has an attractor that is not transitive.

Strange attractors

- The Lorenz attractor.

The Lorenz equations: $$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = x(\rho - z) - y, \\ \dot{z} = xy - \beta z, \end{cases}$$

where σ, ρ, β are parameters. In the case $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, the system has a "strange" attractor.



Strange attractors

- The Hénon attractor.

The Hénon map is a simplified version of the first-return map for the Lorenz system:

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix},$$

where a, b are parameters. In the case $a = 1.4$, $b = 0.3$, the system has a strange attractor.