MATH 614

Dynamical Systems and Chaos

Lecture 26: More on hyperbolic dynamics. Morse-Smale diffeomorphisms.

Chain recurrence

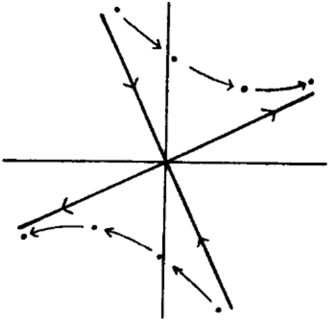
Suppose X is a metric space with a distance function d.
Let $F : X \to X$ be a continuous transformation.

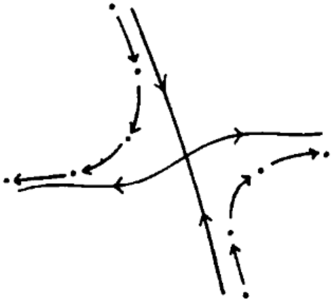
Definition. A point $x \in X$ is **recurrent** for the map F if for any $\varepsilon > 0$ there is an integer $n > 0$ such that $d\big(F^n(x), x\big) < \varepsilon$. The point x is **chain recurrent** for F if, for any $\varepsilon > 0$, there are points $x_0 = x, x_1, x_2, \dots, x_k = x$ and positive integers $n_1, n_2, \dots, n_k$ such that $d\big(F^{n_i}(x_{i-1}), x_i\big) < \varepsilon$ for $1 \le i \le k$.

A sequence $x_0, x_1, \dots, x_k$ is called an ε-**pseudo-orbit** of the map F if $d\big(F(x_{i-1}), x_i\big) < \varepsilon$ for $1 \le i \le k$. The point $x \in X$ is chain recurrent for F if, for any $\varepsilon > 0$, there exists an ε-pseudo-orbit $x_0, x_1, \dots, x_k$ with $x_0 = x_k = x$.

Hyperbolic dynamics

Phase portraits of a linear and a nonlinear two-dimensional maps near a saddle point:





Stable and unstable manifolds

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism and suppose p is a saddle point of F of period m.

Theorem There exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that

(i) $\gamma(0) = p$;
(ii) $\gamma'(0)$ is an unstable eigenvector of $DF^m(p)$;
(iii) $F^{-1}(\gamma) \subset \gamma$;
(iv) $\|F^{-n}(\gamma(t)) - F^{-n}(p)\| \to 0$ as $n \to \infty$.
(v) $\|F^{-n}(x) - F^{-n}(p)\| < \varepsilon$ for all $n \geq 0$, then $x = \gamma(t)$ for some t.

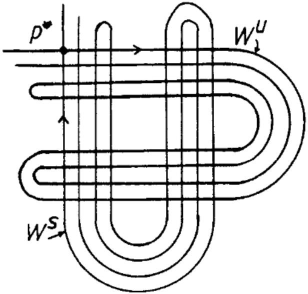
The curve γ is called the **local unstable manifold** of F at p. The **local stable manifold** of F at p is defined as the local unstable manifold of F^{-1} at p.

Homoclinic points

Let $F : X \to X$ be a homeomorphism of a metric space X.

Definition. Suppose $x \in W^s(p) \cap W^u(q)$, where p and q are periodic points of F. Then x is called **heteroclinic** if $p \neq q$ and **homoclinic** if $p = q$.

- Any homoclinic point is chain recurrent.



Morse-Smale diffeomorphisms

Definition. A diffeomorphism $F : X \to X$ is called **Morse-Smale** if

(i) it has only finitely many chain recurrent points,
(ii) every chain recurrent point is periodic,
(iii) every periodic point is hyperbolic,
(iv) all intersections of stable and unstable manifolds of saddle points of F are transversal.

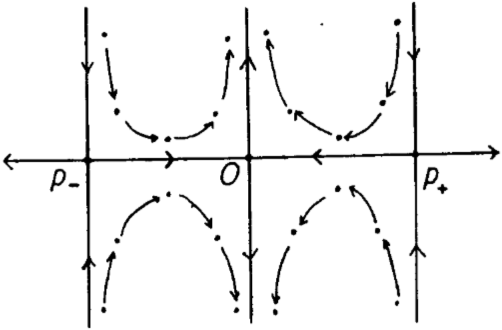
Theorem (Palis) Any Morse-Smale diffeomorphism of a compact surface is C^1-structurally stable.

Example

- $F(x, y) = (x_1, y_1)$, where $x_1 = \dfrac{1}{2}(x + x^3)$,

$y_1 = y \cdot \dfrac{2}{1 + 2x^2}$.

There are three fixed points: $p_+ = (1, 0)$, $p_- = (-1, 0)$ and $O = (0, 0)$. All three are saddle points.

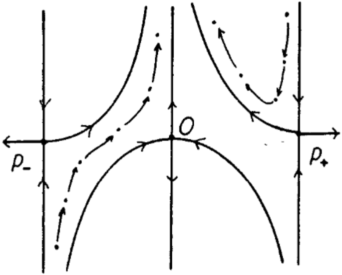


Example

- $F(x, y) = (x_1, y_1)$, where $x_1 = \frac{1}{2}(x + x^3)$,

$y_1 = y \cdot \frac{2}{1 + 2x^2} + \phi(|x|)$, where $\phi(t) > 0$ for $0 < t < 1$ and $\phi(t) = 0$ otherwise.

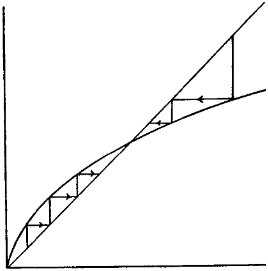
There are still three fixed points: $p_+ = (1, 0)$, $p_- = (-1, 0)$ and $O = (0, 0)$. All three are still saddle points.

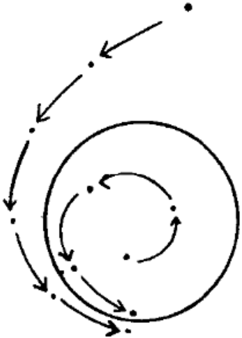


The map F is a Morse-Smale diffeomorphism.

Example

In polar coordinates (r, θ), $F(r, \theta) = (r_1, \theta_1)$,
where $r_1 = 2r - r^3$, $\theta_1 = \theta + 2\pi\omega$.

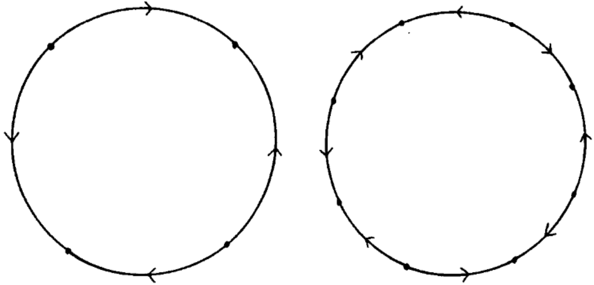




The chain recurrent points are the origin and all points of the invariant circle $r = 1$.

Example

$F(r, \theta) = (r_1, \theta_1)$, where $r_1 = 2r - r^3$, $\theta_1 = \theta + 2\pi(p/q) + \varepsilon \sin(q\omega)$, $p, q \in \mathbb{Z}$ and $\varepsilon > 0$ is small.



The restriction of F to the invariant circle $r = 1$ is a Morse-Smale diffeomorphism of the circle. It follows that F is a Morse-Smale diffeomorphism of the plane.