MATH 614

Dynamical Systems and Chaos

Lecture 34: The Fatou components. The filled Julia set.

The Julia and Fatou sets

Suppose $P : U \to U$ is a holomorphic map, where U is a domain in $\mathbb{C}$, the entire plane $\mathbb{C}$, or the Riemann sphere $\overline{\mathbb{C}}$.

Definition. The **Julia set** $J(P)$ of P is the closure (in U) of the set of repelling periodic points of P. The **Fatou set** $S(P)$ of P is the set of all points $z \in U$ such that the family of iterates $\{P^n\}_{n\geq 1}$ is normal at z.

- $J(P) \cap S(P) = \emptyset$ and $J(P) \cup S(P) = U$.
- $P(J(P)) = J(P)$ and $P^{-1}(J(P)) = J(P)$.
- $P(S(P)) \subset S(P)$ and $P^{-1}(S(P)) = S(P)$.
- If $U \subset \mathbb{C}$ and $\mathbb{C} \setminus U$ contains at least two points, then $S(P) = U$ and $J(P) = \emptyset$.
- If $S(P) \neq \emptyset$, then the Julia set has empty interior.
- If the Julia set is more than one repelling orbit, then it has no isolated points.
- If the Julia set is more than one repelling orbit, then the map P is chaotic on $J(P)$.

The Fatou components

The Fatou set $S(P)$ of a nonconstant holomorphic map $P : U \to U$ is open. Connected components of this set are called the **Fatou components** of P.

- For any Fatou component D of P, the image $P(D)$ is also a Fatou component of P.

- For any Fatou component D of a rational function P there exist integers $k \geq 0$ and $n \geq 1$ such that the Fatou component $P^k(D)$ is invariant under P^n (Sullivan 1986).

- Some transcendental functions P admit a Fatou component D that is a **wandering domain**, i.e., $D, P(D), P^2(D), \ldots$ are disjoint sets.

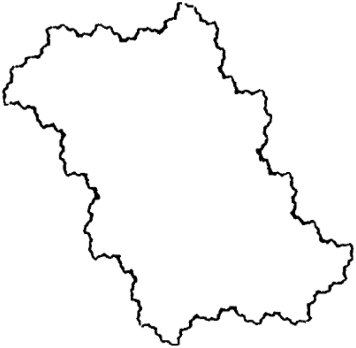
The Fatou components

There are 5 types of invariant Fatou components for a holomorphic map $P : U \to U$:

- **immediate basin of attraction** of an attracting fixed point lying inside the component;
- **attracting petal** of a neutral fixed point lying on the boundary of the component;
- **Siegel disc**: the restriction of P to the component is holomorphically conjugate to a rotation of a disc;
- **Herman ring**: the restriction of P to the component is holomorphically conjugate to a rotation of an annulus;
- **Baker domain**: the iterates of P converge (uniformly on compact subsets of the component) to a constant $z_0 \notin U$ that is an essential singularity of P.

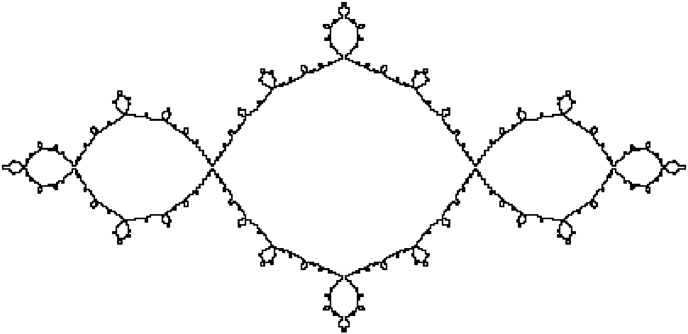
The Baker domains cannot occur for a rational function P.
The Herman rings cannot occur for functions $P : \mathbb{C} \to \mathbb{C}$.

Basin of attraction



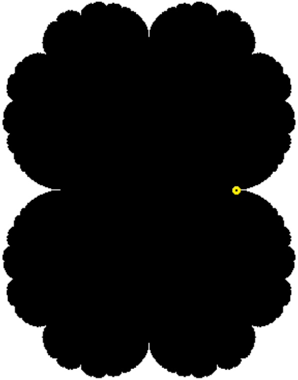
$$P(z) = z^2 + \frac{i}{2}$$

Basins of attraction



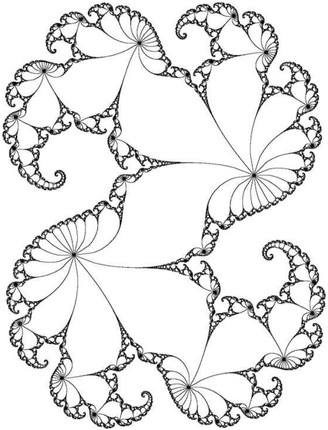
$$P(z) = z^2 - 1$$

Attracting petal



$$P(z) = z^2 + \frac{1}{4}$$

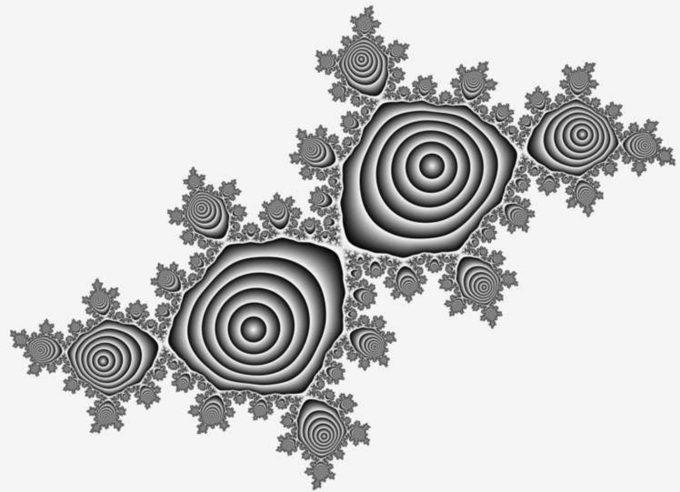
Attracting petals



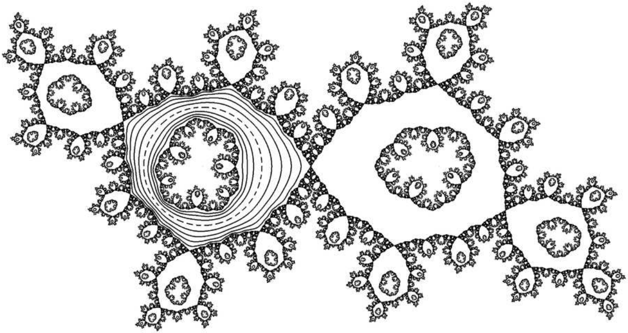
$$P(z) = z^2 + c, \text{ where } c \approx 0.29 + 0.176i.$$

c is chosen on the boundary of the main cardioid of the Mandelbrot set so that P has a fixed point with multiplier $\exp(\frac{2\pi i}{15})$.

Siegel disc



Herman ring



$$P(z) = e^{2\pi i\tau} z^2 \frac{z-4}{1-4z}, \;\; \tau \approx 0.615.$$

The dashed curve is the unit circle $|z| = 1$, which is invariant under P. The restriction of P is an orientation-preserving homeomorphism. τ is chosen so that the rotation number is $(\sqrt{5}-1)/2$.

Polynomial maps

From now on, we assume that P is a polynomial map with $\deg P \geq 2$:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_n \neq 0$, $n \geq 2$. We consider P as a transformation of $\overline{\mathbb{C}}$.

Proposition The point at infinity is a super-attracting fixed point of P.

Proof: Clearly, $P(\infty) = \infty$. To find the derivative $P'(\infty)$, we need to compute the derivative $R'(0)$ of a rational function $R(z) = 1/P(1/z)$. Since $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, it follows that $R(z) = z^n/(a_n + a_{n-1} z + \cdots + a_1 z^{n-1} + a_0 z^n)$. Since $a_n \neq 0$ and $n \geq 2$, we obtain that $R'(0) = 0$.

The filled Julia set

Definition. The **filled Julia set** of the polynomial P, denoted $K(P)$, is the set of all points $z \in \mathbb{C}$ such that the orbit $z, P(z), P^2(z), \dots$ is bounded.

Proposition 1 The complement of $K(P)$ consists of points whose orbits escape to infinity.

Proposition 2 There is $R_0 > 0$ such that the set $\{z \in \mathbb{C} : |z| > R_0\}$ is contained in the Fatou set.

Proposition 3 The Julia set and the filled Julia set are bounded.

Proposition 4 The Julia set is contained in the filled Julia set.

More properties of the filled Julia set

- The filled Julia set is completely invariant: $P(K(P)) \subset K(P)$ and $P^{-1}(K(P)) \subset K(P)$.

- The complement of the filled Julia set is contained in the Fatou set.

- The filled Julia set is closed.

- The filled Julia set is nonempty.

- The interior of the filled Julia set is contained in the Fatou set.