Isospectrality and projective geometries

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In the well-known paper of 1966 M. Kac posed the following question: whether isospectrality of domains in $\mathbb{R}^2$ (i.e. coincidence of spectra for the corresponding Laplace operators) imply that they are isometric. L. Bers stuck the wide-spread label to this problem: "Can one hear the shape of a drum?". In 1992 C. Gordon, D. Webb and S. Wolpert [1] gave the negative answer to the question.

The present text contains rather concise an exposition of certain relation between isospectrality and finite projective geometries; the relation puts an example from [1] into a series of isospectral non-isometric domains and surfaces. Our construction of this series is based on the existence of (spectrally) equivalent and non-conjugate transformation groups whereas projective geometries come forward as a source of such group actions.

Spectrally equivalent non-isomorphic transformation groups have been previously used in algebraic number theory. Certain relation of these groups to the existence of isospectral non-isometric manifolds has been first observed by T. Sunada [2]. His construction was interpreted (and generalized to orbifolds) by P. Berard [3] in terms of what he called transplantation of functions. Finally, the authors of [1] managed to make step from orbifolds to plane domains.

We shall start in our exposition from the simplest, to our knowledge, example of isospectral non-congruent domains on the plane. Let T be a triangle; given fourteen copies of T we build two figures, $F^{(1)}$ and $F^{(2)}$, as shown in the following picture. The construction can be done without overlapping, if the triangle T is acute.

Proposition 1 a) *Laplace operators in the domains $F^{(1)}$ and $F^{(2)}$ have the same spectrum under Dirichlet boundary condition as well as that of Neumann.*

b) *If the angles of T are distinct then polygons $F^{(1)}$ and $F^{(2)}$ are non-congruent.*

Proof. Let the triangle blocks composing $F^{(i)}$, i = 1 or 2, be enumerated as shown in the figure above. Now arbitrary function $f^{(i)}$ defined on $F^{(i)}$ may be represented by a vector-function $f_1^{(i)}, \ldots, f_7^{(i)}$ on T, namely $f_j^{(i)}$ is identified with the restriction of $f^{(i)}$ to the j-th triangle. Using this representation we define two linear operators U_N and U_D, acting from $L_2(F^{(1)})$ to $L_2(F^{(1)})$ by the formulas

$$\begin{pmatrix} f_1^{(2)} \\ f_2^{(2)} \\ f_3^{(2)} \\ f_4^{(2)} \\ f_5^{(2)} \\ f_6^{(2)} \\ f_7^{(2)} \end{pmatrix} = \begin{pmatrix} p & q & p & q & p & q & p \\ q & p & p & q & q & p & p \\ p & p & q & q & p & p & q \\ q & q & q & p & p & p & p \\ p & q & p & p & q & p & q \\ q & p & p & p & p & q & q \\ p & p & q & p & q & q & p \end{pmatrix} \begin{pmatrix} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \\ f_4^{(1)} \\ f_5^{(1)} \\ f_6^{(1)} \\ f_7^{(1)} \end{pmatrix}, \qquad \begin{pmatrix} f_1^{(2)} \\ f_2^{(2)} \\ f_3^{(2)} \\ f_4^{(2)} \\ f_5^{(2)} \\ f_6^{(2)} \\ f_7^{(2)} \end{pmatrix} = \begin{pmatrix} p & -q & p & -q & p & -q & p \\ -q & p & -p & q & -q & p & -p \\ p & -p & q & -q & p & -p & q \\ -q & q & -q & p & -p & p & -p \\ p & -q & p & -p & q & -p & q \\ -q & p & -p & p & -p & q & -q \\ p & -p & q & -p & q & -q & p \end{pmatrix} \begin{pmatrix} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \\ f_4^{(1)} \\ f_5^{(1)} \\ f_6^{(1)} \\ f_7^{(1)} \end{pmatrix}.$$

Here $p = (22 + 12\sqrt{2})^{-1/2}$, $q = -(2 + \sqrt{2})(22 + 12\sqrt{2})^{-1/2}$. Irrespective of the choice of 7×7– matrix in the definition above, operator U_N possesses the property $\Delta U_N f^{(1)} = U_N \Delta f^{(1)}$ if the both sides of this equality are defined. Furthermore, because of the specific design of the matrix in the definition, for arbitrary choice of p and q, operator U_N transforms the domain of Laplace operator in $F^{(1)}$ with Neumann boundary condition into the domain of the corresponding operator in $F^{(2)}$. Finally, the choice of p and q indicated above ensures orthogonality of matrix and hence U_N is unitary in this case. So U_N realizes unitary isomorphism of the Laplace-Neumann operators in the domains $F^{(1)}$ and $F^{(2)}$. Similarly, U_D implements unitary isomorphism of the Laplace- Dirichlet operators in $F^{(1)}$ and $F^{(2)}$.

Now we pass to the claim b). Let the angles of T be distinct and suppose that α is the maximal and γ is the minimal of them. Then in each of the polygons $F^{(1)}$ and $F^{(2)}$ there is the unique angle of the size 4α (maximal angle) and the unique angle of the size γ (minimal angle). For one of the polygons $F^{(i)}$, i = 1 or 2, these two angles are neighbouring and for another one they are not. Hence $F^{(1)}$ and $F^{(2)}$ are non-congruent. ∎

Next we describe the general construction for which the example just considered is the (simplest) particular case. This construction will turn out to be dual (in a sense) for Sunada's quotients.

Let G be a finite group and $g_1, \ldots, g_m$ be its generators. By base figure, corresponding to this set of generators,we call arbitrary flat figure T having segments in its boundary marked by the elements $g_1, \ldots, g_m$ and their inverses (if an element g_i is of order 2 then the marks g_i and g_i^{-1} coincide and are prescribed to the same segment). These segments will be referred to as sides of the base figure T. It is moreover assumed that (a) sides with different marks do not overlap (at most they may have common ends); (b) sides marked by mutually inverse elements g_i and g_i^{-1} have the same length; (c) all sides are endowed independently with some orientation. Base figure may be considered as a polygon with its angles being cut off. The union of this cuts, i.e. the part of the boundary remaining upon deleting all sides, will be called the free boundary.

Further, let $\phi = \{\phi_g\}_{g \in G}$ be an action of G on a finite set M, $\#M = n$. By composed figure, corresponding to an action ϕ and the base figure T, we will call a figure pasted of n figures T_i, $i \in M$, in such a way that (a) each of the figures T_i is isometric to the base figure T by isometry that respects marking and orientation of sides; (b) different figures T_i do not overlap, however they may have common sides and orientations of these common sides should agree; (c) if g is one of the generators $g_1, \dots, g_m$ or their inverses then the side of a figure T_i marked by g coincides with the side of the figure $T_{\phi_g(i)}$ marked by g^{-1}. Composed figure is connected if the action ϕ is transitive. In what follows only transitive actions are considered.

Notice that while base figure is a plane one, composed figure can not in general be realized on (imbedded into) the plane or can be — though not for any choice of the base figure. Nevertheless, composed figure can always be built on an appropriately chosen surface with boundary endowed with Riemannian metric of zero curvature (and perhaps having a finite set of singular points).

Finally, let $\phi^{(1)}$ and $\phi^{(2)}$ be two actions of G on n-element sets $M^{(1)}$ and $M^{(2)}$ respectively. Choosing arbitrarily base figure T one can build composed figures $F^{(1)}$ and $F^{(2)}$ corresponding to T and actons $\phi^{(1)}$, $\phi^{(2)}$.

The construction is completed. Before examining it we recall here in our context some notions of equivalence for transformation groups. Two actions $\phi^{(1)}$ and $\phi^{(2)}$ are called conjugate (or isomorphic) if there exists bijection $\psi : M_1 \to M_2$ such that $\psi \circ \phi_g^{(1)} = \phi_g^{(2)} \circ \psi$ for every $g \in G$. We notice that in the case of conjugate actions $\phi^{(1)}$ and $\phi^{(2)}$ corresponding figures $F^{(1)}$ and $F^{(2)}$ are isometric and moreover there is an isometry that sends base figures composing $F^{(1)}$ onto that of $F^{(2)}$.

Further, given a finite set M we denote by $l_2(M)$ the space of functions on M endowed with the standard scalar product $(f_1, f_2) = \sum_{j \in M} f_1(j)\, \overline{f_2(j)}$. Given an action ϕ of a finite group G on a set M, one gets a unitary representation Φ of the group G in the space $L_2(M)$ defined by the formula $\Phi_g(f) = f \circ \phi_g^{-1}$, $g \in G$. Consider now two representations $\Phi^{(1)}$ and $\Phi^{(2)}$ of the group G, in the spaces $l_2(M^{(1)})$ and $l_2(M^{(2)})$ respectively, generated by the actons $\phi^{(1)}$ and $\phi^{(2)}$. One says that an operator $u : L_2(M^{(1)}) \to L_2(M^{(2)})$ intertwines the representations $\Phi^{(1)}$ and $\Phi^{(2)}$, if $u\Phi_g^{(1)} = \Phi_g^{(2)} u$ for every $g \in G$. If there is a unitary operator intertwining $\Phi^{(1)}$ and $\Phi^{(2)}$ then these representations are called unitarily equivalent while the actions $\phi^{(1)}$ and $\phi^{(2)}$ are called spectrally equivalent.

It turns out that in the case of spectrally equivalent actions $\phi^{(1)}$ and $\phi^{(2)}$, the figures $F^{(1)}$ and $F^{(2)}$ are isospectral (we shall discuss this fact below). Besides, if the actions $\phi^{(1)}$ and $\phi^{(2)}$ are not conjugate then for generic base figure T the figures $F^{(1)}$ and $F^{(2)}$ are non-isometric.

Now let $A = (a(j_2, j_1))$ be an arbitrary $n \times n$–matrix with its rows enumerated by the elements of $M^{(2)}$ and its columns enumerated by the elements of $M^{(1)}$. In each of the spaces $L_2(M^{(i)})$, $i = 1$ or 2, we fix the natural orthonormal basis formed by the functions $\chi_j^{(i)}$, $j \in M^{(i)}$, where $\chi_j^{(i)}(x) = 1$, if $x = j$, and $\chi_j^{(i)}(x) = 0$ otherwise. With respect to these bases the matrix A defines linear operator $u_A : L_2(M^{(1)}) \to L_2(M^{(2)})$.

We shall use the matrix A to introduce one more linear operator as follows. As it has already been described above, each of the figures $F^{(i)}$ ($i = 1$ or 2) is composed of n figures $T_j^{(i)}$, $j \in M^{(i)}$, isometric to the base figure. Hence, arbitrary function $f^{(i)}$ on $F^{(i)}$ can be represented as a set of n functions $f_j^{(i)}$, $j \in M^{(i)}$, defined on T, where the function $f_j^{(i)}$ is identified with the restriction of $f^{(i)}$ to $T_j^{(i)}$. We define operator U_A that sends a function $f^{(1)} = (f_j^{(1)})_{j\in M^{(1)}}$, defined on $F^{(1)}$, into the function $f^{(2)} = (f_j^{(2)})_{j\in M^{(2)}}$, defined on $F^{(2)}$, according to the formula

$$f_{j_2}^{(2)} = \sum_{j_1 \in M^{(1)}} a(j_2, j_1)\, f_{j_1}^{(1)}.$$

U_A is called a transplantation operator. We will consider U_A as an operator acting from $L_2(F^{(1)})$ into $L_2(F^{(2)})$.

Let $\Delta^{(1)}$ be the Laplace operator on the figure $F^{(1)}$ defined by Neumann (or Dirichlet) boundary condition and $\Delta^{(2)}$ be the corresponding operator on $F^{(2)}$. For the very existence of the Laplace-Neumann operator it is necessary that the free boundary of the base figure be piecewise smooth.

Definition 1 *Operator U_A intertwines (corresponding) Laplace operators $\Delta^{(1)}$ and $\Delta^{(2)}$ if for every $f \in L_2(F^{(1)})$ from the domain of $\Delta^{(1)}$, the function $U_A f$ belongs to the domain of $\Delta^{(2)}$ and moreover*

$$\Delta^{(2)}\, U_A f = U_A\, \Delta^{(1)} f.$$

We remark here that this equality surely holds iff its both sides are defined. It follows from the specific form of the transplantation operator and also from the fact that Laplace operator is equivariant with respect to isometries. Hence the only nontrivial condition in the definition of intertwining property is that U_A transforms the domain of $\Delta^{(1)}$ into that of $\Delta^{(2)}$. If the operator U_A is unitary (obviously the necessary and sufficient condition for this property is that A be unitary) then it implements isomorphism (unitary equivalence) of operators $\Delta^{(1)}$ and $\Delta^{(2)}$ and hence they have common spectrum.

Before stating the main condition ensuring that transplantation operator intertwines Laplace operators, it is convenient to impose a restriction on geometrical properties of composed figures. We shall say that the figure (surface) $F^{(i)}$ satisfies condition I if the corner points of the composing figures (blocks) $T_j^{(i)}$ (i.e. common ends of the adjacent sides of these figures) are not interior points for $F^{(i)}$. If the condition I is satisfied then any interior point of $F^{(i)}$ is either an interior point of some $T_j^{(i)}$ or an interior point of some joint side of the blocks. If the condition I is not satisfied, one can seek its fulfilment by a slight modification of the base block —it is sufficient to cut one angle (may be several angles) off T.

Proposition 2 *Assume that figures $F^{(1)}$ and $F^{(2)}$ satisfy the condition I. Then transplantation operator U_A intertwines Laplace operators on this figures, defined by Neumann boundary* *conditions iff the operator u_A intertwines representations $\Phi^{(1)}$ and $\Phi^{(2)}$. In particular, under the assumption made, spectral equivalence of actions $\phi^{(1)}$ and $\phi^{(2)}$ implies coincidence of spectra for the Laplace-Neumann operators.*

There is a similar statement about Laplace operators defined by Dirichlet boundary conditions. Instead of representations $\Phi^{(1)}$ and $\Phi^{(2)}$, it deals with other representations built in a certain way by actions $\phi^{(1)}$ and $\phi^{(2)}$.

Besides isospectrality of $F^{(1)}$ and $F^{(2)}$, one needs these figures to be also nonisometric. It is necessary to take care of this property since even for nonconjugate actions $\phi^{(1)}$ and $\phi^{(2)}$ due to unsatisfactory choice of the base block it may happen that the corresponding composite figures are isometric. We shall confine ourselves to the following existence claim.

Proposition 3 *For any finite group G with a fixed set of generators, one can choose such a plane domain T with piesewise smooth boundary that (a) the composite figure, built by T and arbitrary transitive action of G on a finite set, satisfies condition I and (b) the corresponding composite figures, built by a pair of nonconjugate actions, are non-isometric.*

So the problem of searching for isospectral non-isometric domains or surfaces is (partially) reduced to producing spectrally equivalent and non-conjugate actions of finite groups (compare with the following von Neumann theorem: spectral equivalence implies conjugacy for ergodic dynamical systems of pure point spectrum). The main source of such actions for us will be finite projective geometries (in particular, Galois geometries).

Now we consider arbitrary Galois geometry, i.e. finite dimensional projective space $M^{(1)}$ over a finite field. Let k ($k \geq 2$) be its dimension and r be the order of the field. Denote by $M^{(2)}$ the set of hyperplanes of codimension 1 in $M^{(1)}$ (the set of lines, in the case of projective plane). Obviously $\# M^{(1)} = \# M^{(2)}$. Denote by G the group of projective transformations of the geometry $M(1)$, and by $\phi^{(1)}$ and $\phi^{(2)}$ the natural actions of the group G on the sets $M^{(1)}$ and $M^{(2)}$ respectively.

Proposition 4 *The actions $\phi^{(1)}$ and $\phi^{(2)}$ are spectrally equivalent and non-conjugate.*

PROOF. Let $A = (a(j_2, j_1))$, $j_1 \in M^{(1)}$, $j_2 \in M^{(2)}$ be the incidence matrix for the geometry considered, i.e. $a(j_2, j_1) = 1$, if the point j_1 belongs to the hyperplane j_2, and $a(j_2, j_1) = 0$ — otherwise. Denote by B the matrix that is obtained from A by changing zeroes to ones. The representations $\Phi^{(1)}$ and $\Phi^{(2)}$ of the group G, generated by the actions $\phi^{(1)}$ and $\phi^{(2)}$ respectively, are intertwined by operator u_B and by operator u_A as well. Hence for any reals p and q operator $u_{(q-p)A+pB} = (q-p)u_A + pu_B$ intertwines also the representations $\Phi^{(1)}$ and $\Phi^{(2)}$. Straightforward computation shows that $((q-p)u_A + pu_B)\,((q-p)u_A + pu_B)^* = (n_1q^2 + (n-n_1)p^2)I + (n_2q^2 + 2(n_2-n_1)pq + (n-2n_1+n_2)p^2)(J-I)$, where n is the number of points in the geometry $M^{(1)}$, n_1 is the number of points in the hyperplane of codimension 1, n_2 is the number of points in the hyperplane of codimension 2, I is the identity $n \times n$-matrix and J is the $n \times n$-matrix with all entries equal to 1. Remark that the square polynomial $n_2x^2 + 2(n_2-n_1)x + (n-2n_1+n_2)$ has all its roots real. Indeed, $n = 1 + r + \cdots + r^k$, $n_1 = 1 + r + \cdots + r^{k-1}$, $n_2 = 1 + r + \cdots + r^{k-2}$, hence $2(n_1 - n_2) = 2r^{k-1}$, $n - 2n_1 + n_2 = r^k - r^{k-1}$, and the discriminant of the polynomial above is equal to $(2r^{k-1})^2 - 4(r^k - r^{k-1})(1 + r + \cdots + r^{k-2}) = 4(r^{k-1})^2 - 4r^{k-1}(r^{k-1} - 1) = 4r^{k-1} > 0$. It follows that the reals p, q can be chosen in such a way that the relations $n_2q^2 + 2(n_2 - n_1)pq + (n - 2n_1 + n_2)p^2 = 0$, $n_1q^2 + (n - n_1)p^2 = 1$ hold. Under this choice the operator $u_{(q-p)A+pB}$ is unitary.

Thus the actions $\phi^{(1)}$ and $\phi^{(2)}$ are spectrally equivalent. To establish their non-conjugacy it is sufficient to show that for any $j_1 \in M^{(1)}$ and $j_2 \in M^{(2)}$ there is a projective transformation $g \in G$ that fixes hyperplane j_2 and at the same time "shifts" the point j_1. Every Galois geometry obviously possesses this property. ∎

It remains now to choose the generators of G and one can obtain pairs of isospectral non-isometrac plane domains (surfaces with boundary and flat metric, to be precise). The set of generators can be chosen by different ways and this leads to different examples. Taking into account that there are infinitely many different Galois geometries we arrive at the following result.

Theorem 5 *There exists infinite sequence of pairs of isospectral non-isometric two-dimensional manifolds with boundary having flat Riemannian metric.*

This claim deserves some comments and elucidations. It is certainly sufficient to have only one pair of spectrally equivalent and non-conjugate actons of a finite group in order to get a rich family of pairs of isospectral non-isometric surfaces that are obtained (one from another) by deformation of the main block. The theorem states the existence of infinitely many pairs such that none of them can be obtained from another one by the deformation mentioned above.

Now we shall describe some improvements of the isospectrality construction proposed here that enable one to generalize theorem 5 in several directions. Our first remark concerns a transition from two-dimensional case to multidimensional one. To this end we shall take certain bounded domain T in a (finite-dimensional) Euclidean space as a base figure. Its sides (marked by generators of a finite group G and their inverses) are now assumed to be non-overlapping flat pieces of the boundary; it is more appropriate to use the term "faces" for these pieces. The faces marked by inverse elements of G should be isometric. Moreover, given generator $g \in G$ from the fixed set of generators we choose some isometry χ_g of the face marked by g onto the face marked by g^{-1} (in the two-dimensional case this choice reduces to some orientation of sides); if the generator g is of order 2, the isometry should be identity. Now given action χ of G on a finite set M we build the manifold F composed by copies T_i, $i \in M$, of the base figure T as follows: for every generator $g \in G$ and every $i \in M$ we identify the face of T_i marked by g with the face of $T_{\phi_g(i)}$ marked by g^{-1}, by means of χ_g. For two composite manifolds built like that by two actions of G, one can define transplantation operator just as in two-dimensional case. Propositions 2 and 3, as well as their proofs, remain valid in this situation. As a result we arrive at the following generalization of theorem 5: surfaces with flat metric in the claim can be changed to n-dimensional ($n \geq 2$) manifolds with Euclidean metric.

Next improvement introduces base blocks with arbitrary Riemannian (instead of flat) metric into our construction. Let the base block T now be a compact manifold with boundary endowed with smooth Riemannian metric. One chooses non-overlapping smooth pieces of boundary as the faces of T. The construction of composite manifolds is done similarly to the n-dimensional Euclidean case. The only difficulty here is that the Riemannian metric will not necessary be smooth since singularities of metric may arise at the places where different copies of T are pasted together. We overcome this circumstance imposing additional condition on the base block T. Let $g_1, \ldots, g_m$ be the generators of G that (together with their inverses) are labels of T's faces. Assembling of the base blocks (to produce composed manifolds) is done by means of mappings $\chi_1, \ldots, \chi_m$, where χ_i is an isometry of the face marked by g_i, onto the face marked by g_i^{-1}. Our assumption is that the Riemannian metrics on two copies T_1 and T_2 of the base block T (pasted together by means of some χ_i) can be smoothly extended to the joining seam.

Under the condition just introduced, Laplace operator on composite manifold is defined and proposition 2 still holds (with minor modifications in the proof). This condition is satisfied in the particular important case when the base block is a domain on the surface of constant curvature and all the sides are geodesic segments or, more generally, when the base block T is a domain in the space of constant curvature with completely geodesic pieces of the boundary as faces of T. Proposition 3 holds in this case and we get therefore another generalization of Theorem 5 that deals with Riemannian metric of constant curvature instead of flat metric.

Another possibility of improvement for the theorem 5 concerns certain revision of boundary conditions. Isospectrality there (i.e. in the theorem 5) is meant as coincidence of spectra for Laplace-Neumann operators. It turns out, however, that in the proof of proposition 2 one does not really use specific choice of boundary conditions on the free boundary of composite manifold X, i.e. on the union of free boundaries of composing blocks. It is only used that for all copies of the base blocks (composing X), the boundary conditions are the same. Hence if one changes the boundary conditions on some pieces of the free boundary so that this change respects the rule of pasting blocks together, then the proposition 2 will still be valid. So we get that the transplantation operator intertwining Laplace-Neumann operators for the composite manifolds with nonvoid free boundary simultaneously intertwines different pairs of Laplace operators (on these manifolds) corresponding to the mixed (partially Neumann and partially Dirichlet) boundary conditions. In particular, if the whole boundary of two composite manifolds is free (this is the case, for example, if there are no involutive generators) then the transplantation operator intertwining Laplace-Neumann operators, also intertwines Laplace-Dirichlet operators. This enables one to extend the notion of isospectrality in theorem 5 so that it will include coincidence of spectra both for Laplace-Neumann and Laplace-Dirichlet operators.

Finally, we mention an improvement concerning isospectral multiplicity, i.e. the question about maximal number of pairwise non-isometric isospectral manyfolds (with boundary). Let $\phi^{(1)}$ and $\phi^{(2)}$ be spectrally equivalent and non-conjugate actions of a finite group G. Then given integer $k \geq 2$ for the group G^k there are 2^k spectrally equivalent and non-conjugate actions of the form $\phi^{(i_1)} \times \phi^{(i_2)} \times \ldots \times \phi^{(i_k)}$, where each of the superscripts is either 1 or 2. Choosing (by virtue of proposition 3) an appropriate base figure, one can build by these actions 2^k pairwise non-isometric isospectral manifolds.

We summarize the remarks to theorem 5 given above in the following

Theorem 6 *For every integers $l, n \geq 2$ and arbitrary real R there exists infinitely many l-tuples of isospectral non-isometric n-dimensional manifolds with boundary, having Riemannian metric of constant curvature R.*

Next we shall give certain list of spectrally equivalent non-conjugate actions (for the motivation of this list see below). Each time a group of projective transformations acts: either this is the case of finite geometry of order 7 (projective plane over $\mathbb{F}_2$), geometry of order 13 (projective plane over $\mathbb{F}_3$), or of order 15 (3-dimensional projective space over $\mathbb{F}_2$). In each of these groups there are invilutive (i.e. of order 2) generators a, b, c. Permutations a_1, b_1 c_1 define the action of G on points of projective geometry, while permutations a_2, b_2, c_2 define the action of G on lines and on planes (in the case of geometry of order 15).

Item 1 in the list corresponds to the example given at the very beginning, item 2 corresponds to the example found in [1].

(1) $a_1 = (1,2)(4,7)$, $b_1 = (1,6)(2,5)$, $c_1 = (1,4)(3,6)$; $a_2 = b_1$, $b_2 = a_1$, $c_2 = c_1$.

(2) $a_1 = (1,5)(3,7)$, $b_1 = (1,2)(5,6)$, $c_1 = (1,4)(3,6)$; $a_2 = (4,5)(6,7)$, $b_2 = b_1$, $c_2 = c_1$.

(3) $a_1 = (2,6)(3,7)$, $b_1 = (1,2)(5,6)$, $c_1 = (1,4)(3,6)$; $a_2 = (4,6)(5,7)$, $b_2 = b_1$, $c_2 = c_1$.

(4) $a_1 = (1,8)(2,6)(3,13)(7,11)$, $b_1 = (1,5)(3,8)(4,11)(10,13)$, $c_1 = (6,7)(8,11)(9,13)(10,12)$; $a_2 = (1,2)(5,13)(6,10)(8,12)$, $b_2 = b_1$, $c_2 = c_1$.

(5) $a_1 = (1,8)(2,7)(4,12)(6,11)$, $b_1 = (1,5)(3,8)(4,11)(10,13)$, $c_1 = (6,7)(8,11)(9,13)(10,12)$; $a_2 = (1,2)(5,12)(7,9)(8,13)$, $b_2 = b_1$, $c_2 = c_1$.

(6) $a_1 = (1,8)(2,13)(3,6)(5,12)$, $b_1 = (1,5)(3,8)(4,11)(10,13)$, $c_1 = (6,7)(8,11)(9,13)(10,12)$; $a_2 = (1,8)(2,12)(4,7)(5,13)$, $b_2 = b_1$, $c_2 = c_1$.

(7) $a_1 = (1,9)(2,5)(4,13)(7,12)$, $b_1 = (1,5)(3,8)(4,11)(10,13)$, $c_1 = (6,7)(8,11)(9,13)(10,12)$; $a_2 = (2,7)(3,6)(4,5)(8,9)$, $b_2 = b_1$, $c_2 = c_1$.

(8) $a_1 = (1,13)(2,8)(3,6)(7,9)$, $b_1 = (1,5)(3,8)(4,11)(10,13)$, $c_1 = (6,7)(8,11)(9,13)(10,12)$; $a_2 = (2,4)(5,13)(6,11)(7,12)$, $b_2 = b_1$, $c_2 = c_1$.

(9) $a_1 = (3,4)(5,8)(6,9)(7,10)$, $b_1 = (1,8)(2,6)(3,13)(7,11)$, $c_1 = (6,7)(8,11)(9,13)(10,12)$; $a_2 = (2,11)(3,12)(4,13)(9,10)$, $b_2 = (1,2)(5,13)(6,10)(8,12)$, $c_2 = c_1$.

(10) $a_1 = (3,4)(5,11)(6,12)(7,13)$, $b_1 = (1,8)(2,6)(3,13)(7,11)$, $c_1 = (6,7)(8,11)(9,13)(10,12)$; $a_2 = (2,8)(3,10)(4,9)(12,13)$, $b_2 = (1,2)(5,13)(6,10)(8,12)$, $c_2 = c_1$.

(11) $a_1 = (1,9)(3,12)(4,6)(5,11)$, $b_1 = (1,8)(2,6)(3,13)(7,11)$, $c_1 = (6,7)(8,11)(9,13)(10,12)$; $a_2 = (2,9)(3,10)(4,8)(5,7)$, $b_2 = (1,2)(5,13)(6,10)(8,12)$, $c_2 = c_1$.

(12) $a_1 = (1,13)(3,10)(4,7)(5,8)$, $b_1 = (1,8)(2,6)(3,13)(7,11)$, $c_1 = (6,7)(8,11)(9,13)(10,12)$; $a_2 = (2,13)(3,12)(4,11)(5,6)$, $b_2 = (1,2)(5,13)(6,10)(8,12)$, $c_2 = c_1$.

(13) $a_1 = (1,10)(3,8)(4,15)(6,13)$, $b_1 = (1,4)(3,6)(9,12)(11,14)$, $c_1 = (1,8)(2,4)(3,12)(5,10)(7,14)(11,13)$; $a_2 = (1,12)(2,15)(5,8)(6,11)$, $b_2 = b_1$, $c_2 = c_1$.

(14) $a_1 = (1,6)(2,5)(8,15)(11,12)$, $b_1 = (1,4)(3,6)(9,12)(11,14)$, $c_1 = (1,8)(2,4)(3,12)(5,10)(7,14)(11,13)$; $a_2 = (1,10)(2,9)(4,15)(7,12)$, $b_2 = b_1$, $c_2 = c_1$.

(15) $a_1 = (1,2)(5,6)(8,11)(12,15)$, $b_1 = (1,12)(3,14)(4,9)(6,11)$, $c_1 = (1,8)(2,4)(3,12)(5,10)(7,14)(11,13)$; $a_2 = (1,10)(2,9)(5,14)(6,13)$, $b_2 = (1,4)(3,6)(8,13)(10,15)$, $c_2 = c_1$.

(16) $a_1 = (1,13)(3,15)(4,8)(6,10)$, $b_1 = (1,12)(3,14)(4,9)(6,11)$, $c_1 = (1,8)(2,4)(3,12)(5,10)(7,14)(11,13)$; $a_2 = (4,9)(5,8)(6,11)(7,10)$, $b_2 = (1,4)(3,6)(8,13)(10,15)$, $c_2 = c_1$.

The isospectral figures corresponding to these actions, except for example 10, can be realized in the plane, i.e. for an appopriate choice of the base figure both of the composite figures are embedded into Euclidean plane as non-congruent domains. Moreover, in each of these cases one can take certain triangle as the base figure; corresponding domains are shown on the pictures at the end of the text (α, β, γ with its values indicated under the pictures are the angles opposite to the sides a, b, c respectively).

According to proposition 2, Laplace-Neumann operators on composed figures built by any pair of actions in examples 1–16, have common spectra. As regards Dirichlet case, the sufficient condition for unitary isomorphism of Laplace operators does not reduce to spectral equivalence of actions and hence requires a verification. This has been done and it turned out that for each of the examples above the corresponding pair of Laplace-Dirichlet operators have the same spectrum.

Given action of a finite group G (with fixed set of generations) on a finite set M one can build a graph Γ with M as the set of vertices as follows: two vertices are joined by an edge if one of them is sent to another one by some generator. The number of edges joining two vertices is equal to the number of such generators. The actions from our list possess the following property D: corresponding graph Γ is a tree. Actions of projective origin with such a property exist only for projective geometries of order 7, 13 or 15. As the computer checking shows, all of them are given by examples 1–16.

Property D ensures that the composite figure is simply connected and thus it is related to the possibility of plane realization of the composite figure. As regards the embedding to Lobachevski plane, condition D is sufficient. Indeed, for this purpose simply take a polygon with small angles as a base figure. If the angles are rationally independent then the composite figures corresponding to non-conjugate actions are non-isometric. Thus each of the examples leads to a pair of polygons in Lobachevski plane, which are isospectral and non-isometric. Moreover, if the angles are sufficiently small then one can get convex domains (for example 2 this has been noticed by Gordon and Webb [4]).

Concluding remarks and acknowledgements. There is a cohomological approach (suggested by V. M. Alexeyev) to the proof of von Neumann theorem about measure preserving conjugacy of spectrally equivalent ergodic (probability measure preserving) actions of abelian groups having pure point spectrum. This approach naturally gives rise to the non-abelian version of the relation between isomorphism and spectral equivalence. The example of spectrally equivalent non-conjugate actions, related to the 7-point projective plane, was known to one of the authors for a long time (since 60-ies); this example is based on classical results about finite geometries, cf., for instance, “Group theory” by M. Hall.

The observation concerning the role of projective geometries in the isospectrality problems and the basic construction presented here were made in 1996. Corresponding results (in the two-dimensional case) were announced at the conference on dynamical systems and ergodic theory held at IESIMP during 1996–97 academic year. One of the authors (A.M.S.) acknowledges the hospitality of the Erwin Schrodinger Institute. The research was partially supported also by RFBR and INTAS.

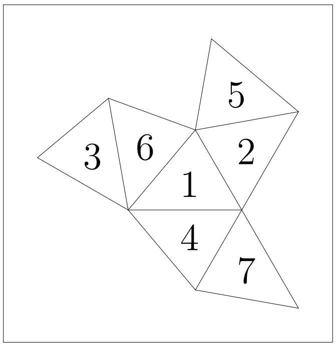
References

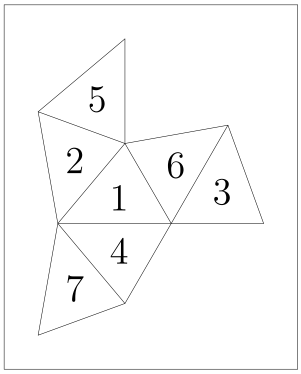
[1] Gordon C., Webb D., Wolpert S. *Isospectral plane domains and surfaces via Riemannian orbifolds* // Invent. Math. **110** (1992), 1. 1–22

[2] Sunada T. *Riemannian coverings and isospectral manifolds* // Ann. of Math. (2) **121** (1985), 1. 169–186

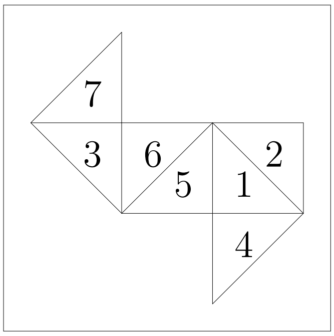
[3] Berard P. *Transplantation et isospectralite I* // Math. Ann. **292** (1992), 3. 547–559

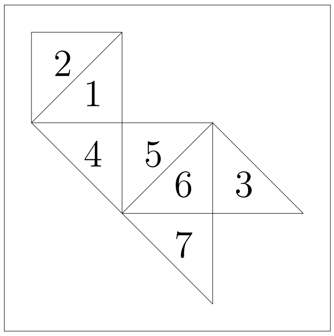
[4] Gordon C., Webb D. *Isospectral convex domains in the hyperbolic plane* // Proc. Amer. Math. Soc. **120** (1994), 3, 981–983



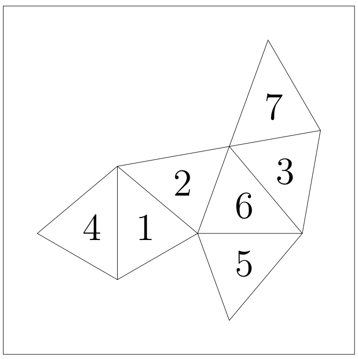


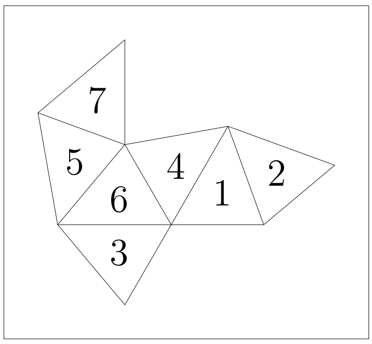
Example 1: $\alpha = 50^o$, $\beta = 60^o$, $\gamma = 70^o$



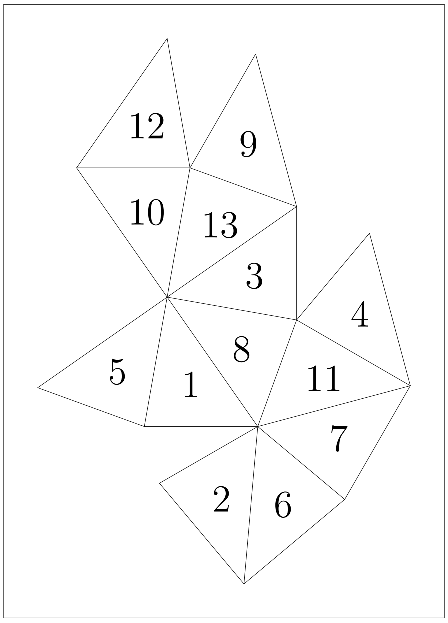


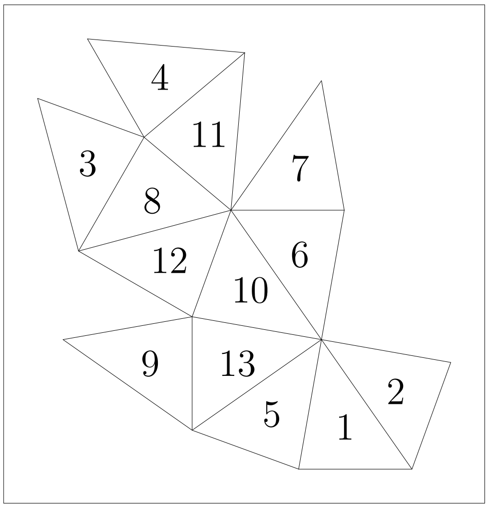
Example 2: $\alpha = 45^o$, $\beta = 90^o$, $\gamma = 45^o$



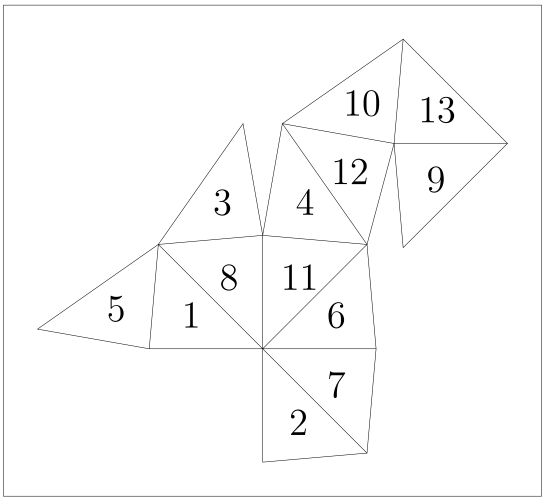


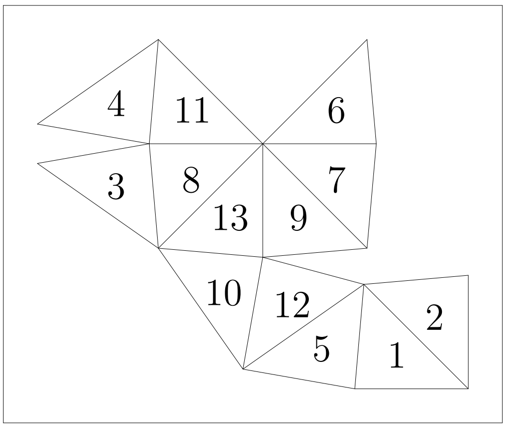
Example 3: $\alpha = 50^o$, $\beta = 60^o$, $\gamma = 70^o$



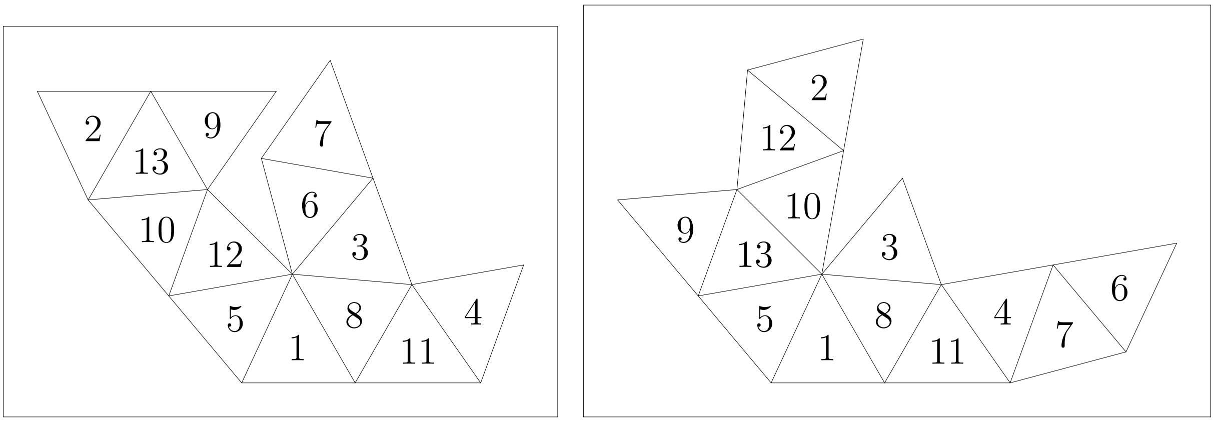


Example 4: $\alpha = 80^o$, $\beta = 55^o$, $\gamma = 45^o$

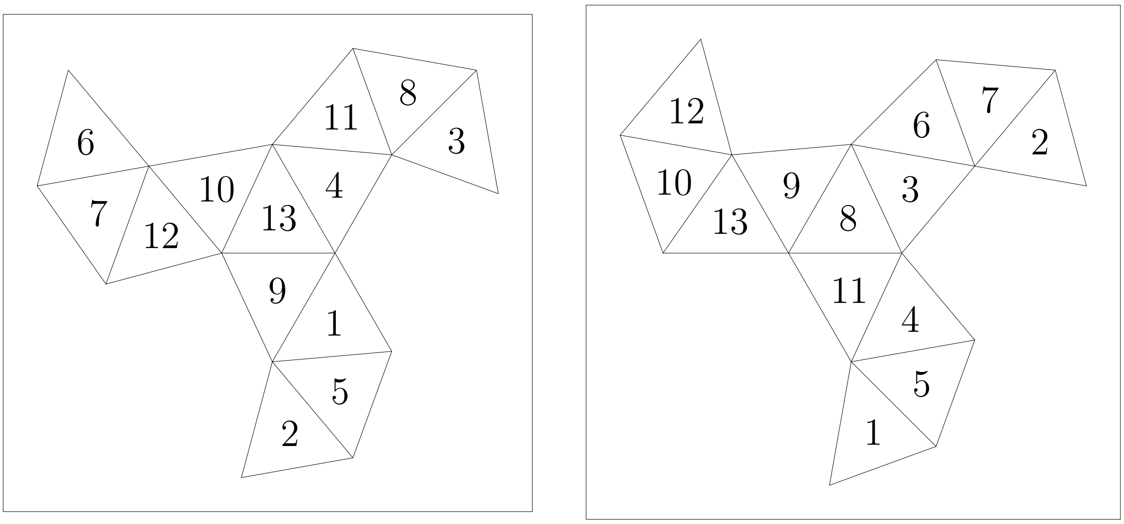




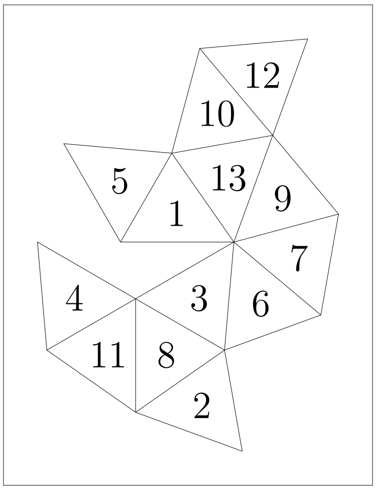
Example 5: $\alpha = 85^o$, $\beta = 45^o$, $\gamma = 50^o$

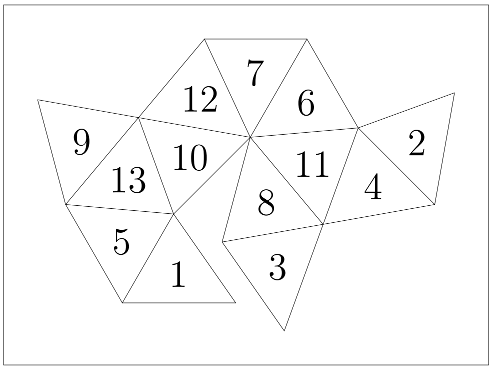


Example 6: $\alpha = 65^o$, $\beta = 60^o$, $\gamma = 55^o$

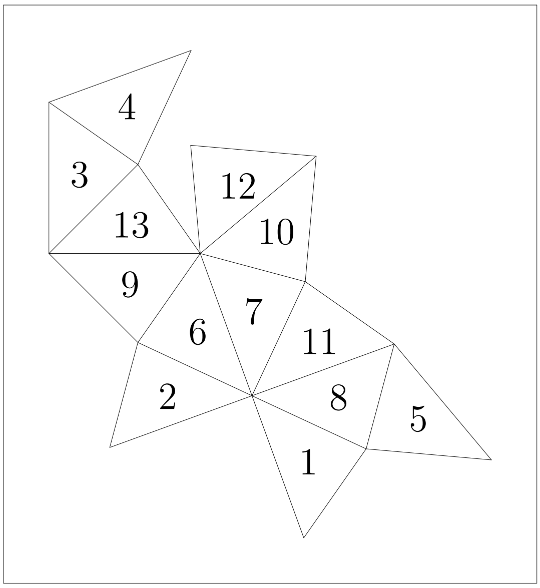


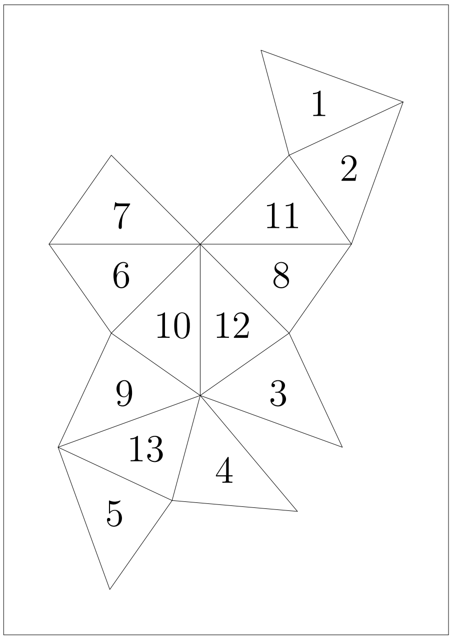
Example 7: $\alpha = 65^o$, $\beta = 60^o$, $\gamma = 55^o$



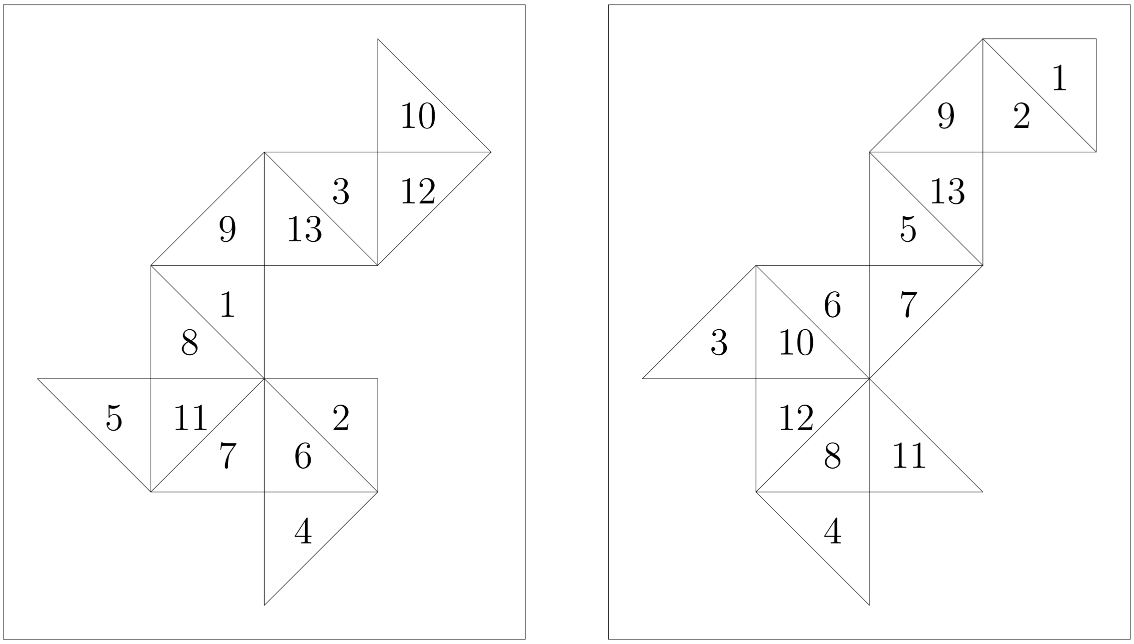


Example 8: $\alpha = 60^o$, $\beta = 55^o$, $\gamma = 65^o$

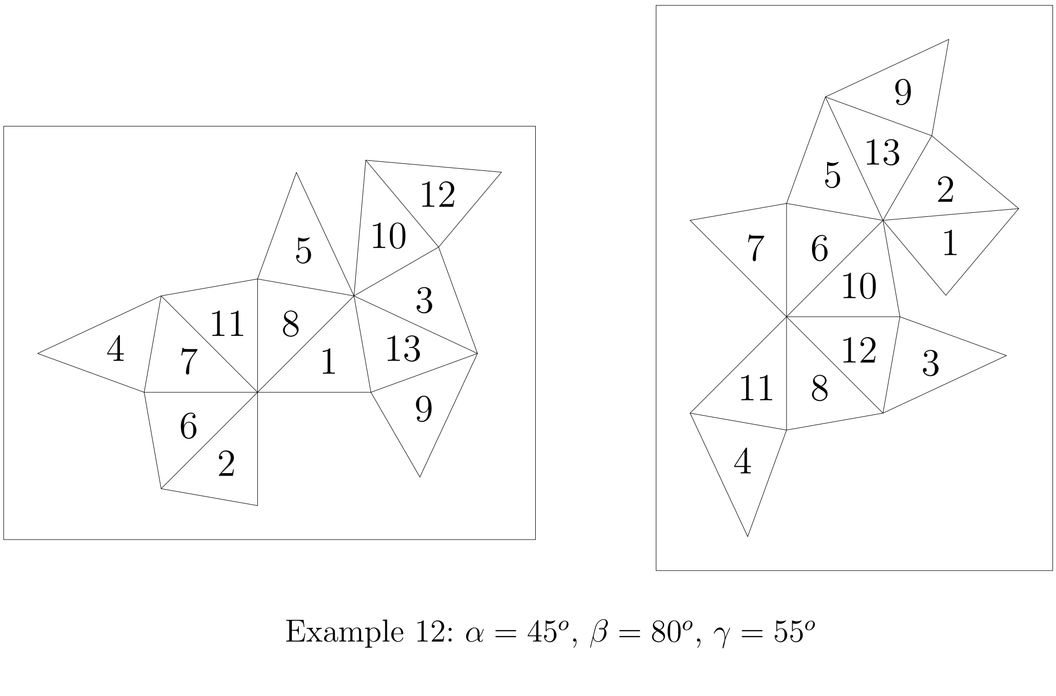




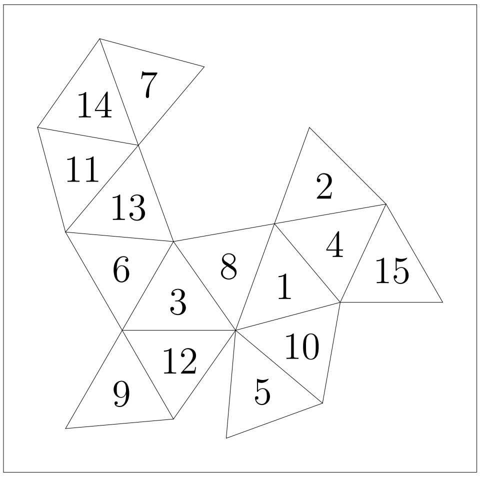
Example 9: $\alpha = 45^o$, $\beta = 55^o$, $\gamma = 80^o$

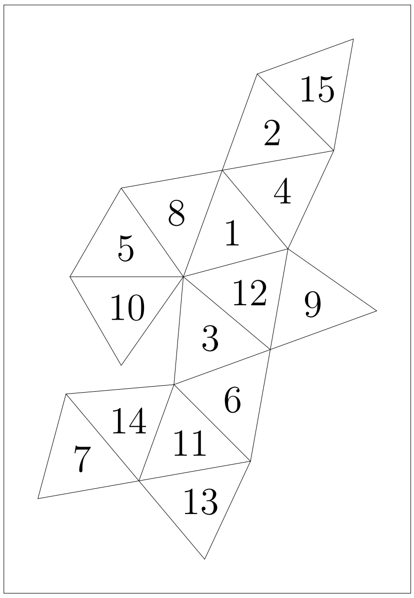


Example 11: $\alpha = 45^o$, $\beta = 90^o$, $\gamma = 45^o$

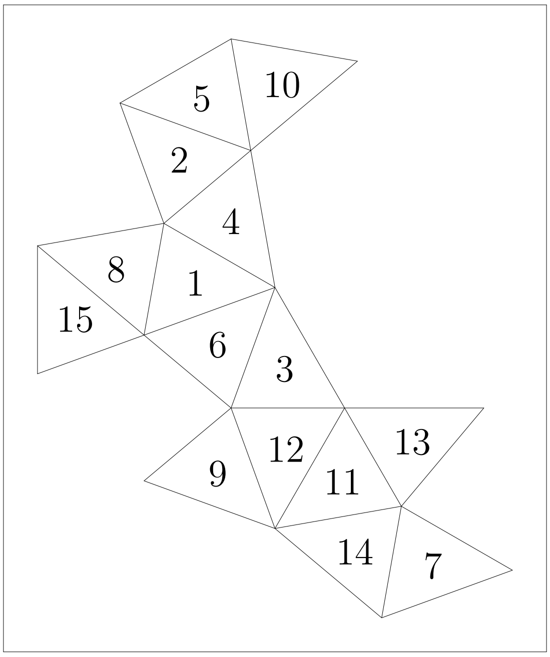


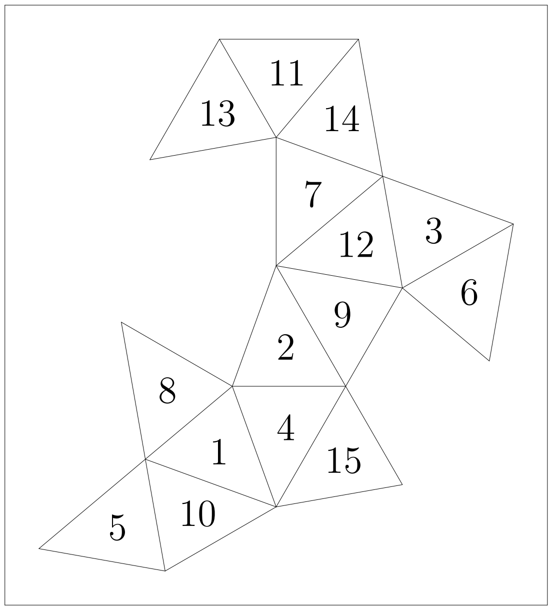
Example 12: $\alpha = 45^o$, $\beta = 80^o$, $\gamma = 55^o$



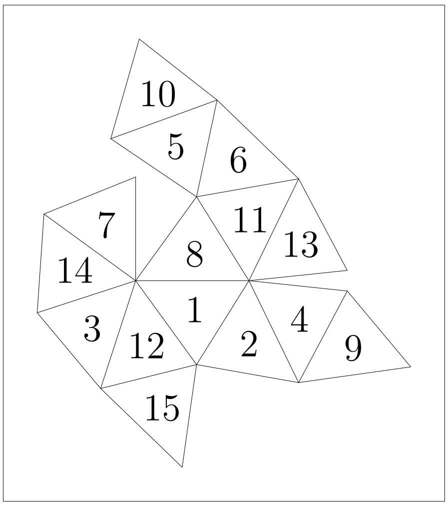


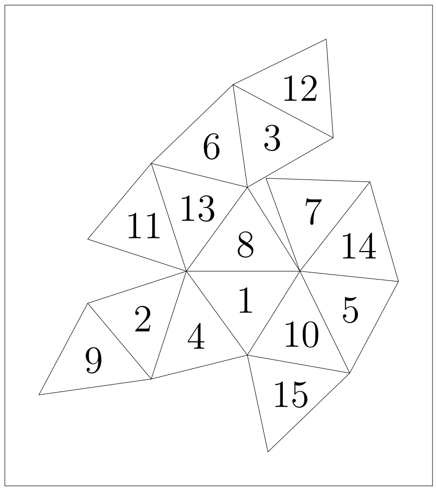
Example 13: $\alpha = 60^o$, $\beta = 55^o$, $\gamma = 65^o$



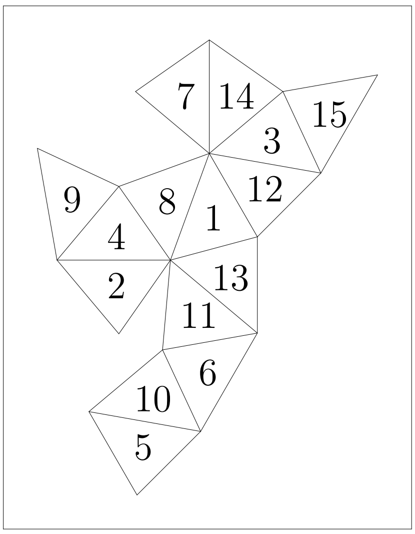


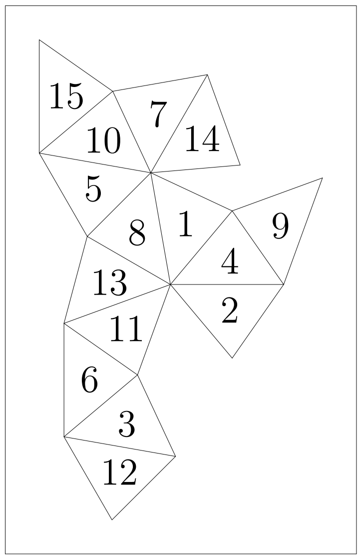
Example 14: $\alpha = 70^o$, $\beta = 60^o$, $\gamma = 50^o$





Example 15: $\alpha = 54^o$, $\beta = 58^o$, $\gamma = 68^o$





Example 16: $\alpha = 50^o$, $\beta = 55^o$, $\gamma = 75^o$