Journal Title: Sbornik. Mathematics

Volume: 188
Issue: 3
Month/Year: 1997
Pages: 389-434

Article Author: Ya. B. Vorobets

Article Title: Ergodicity of billiards in polygons

Status: Faculty
Phone: 845-0849
E-mail: yvorobet@math.tamu.edu

Name: Vorobets, Yaroslav

Pickup at Evans

Address:
MS 3368
College Station, TX 77843-3368

Sbornik: Mathematics **188**:3 389–434 ©1997 RAS(DoM) and LMS
Matematicheskiĭ Sbornik **188**:3 65–112 UDC 517.987.5

Ergodicity of billiards in polygons

Ya. B. Vorobets

Abstract. In the space of all polygons, a topologically massive subset consisting of polygons with ergodic billiard flows is explicitly described. The elements of this set have a specified order of approximation by rational polygons.

As intermediate results, constructive versions of the ergodic theorem for the billiard in a rational polygon and for the geodesic flow on a surface with flat structure, and also a constructive quadratic estimate for the growth of the number of saddle connections (singular trajectories) in a flat structure, are proved.

Bibliography: 6 titles.

1. Introduction

The *billiard* in a plane domain Q with piecewise smooth boundary is the dynamical system describing the frictionless motion in Q of a point-like ball rebounding at the boundary of Q by the law 'the angle of incidence is equal to the angle of reflection'. Usually, one considers the billiard flow on the level set of energy corresponding to the motion with unit velocity, so that one can set the phase space of the flow to be $Q \times S^1$ (here S^1 is the circle of unit velocities) with identification of elements (x, v_1) and (x, v_2) such that x is a boundary point of Q and v_1 and v_2 are vectors symmetric with respect to the tangent line to ∂Q at x. The billiard preserves the natural measure $\mu \times \lambda$ (where μ and λ are Lebesgue measures on Q and S^1, respectively) on the phase space.

Definition 1.1. We say that the billiard in a domain Q is *ergodic* if each measurable subset of the phase space that is invariant with respect to the billiard flow is either of measure zero or of full measure.

The subject of the present paper is the billiard flows in polygonal domains of general form. The reflection condition for a billiard looks particularly simple in such domains, however, billiards in polygons are, with a few exceptions, dynamical systems with singularities due to the presence of corners. The study of the ergodic properties of the flow turns out to be a complicated problem for that reason. The only general results in this case are those obtained for the billiards in so-called rational polygons.

For each polygon Q we denote by $G(Q)$ the group of orthogonal operators generated by the linear parts of the symmetries with respect to the sides of Q.

AMS 1991 *Mathematics Subject Classification.* Primary 58F11, 58F17.

Definition 1.2. We say that a polygon Q is *rational* if $G(Q)$ is a finite group. An equivalent condition requires that the angle between any two sides of the polygon (not necessarily adjacent or lying in the same component of the boundary) is (rationally) commensurable with π.

The phase space of the billiard in a rational polygon Q is foliated by invariant surfaces $Q \times G(Q)v$, $v \in S^1$ (see [1]). As shown by Kerckhoff, Masur, and Smillie [2], the restriction of the billiard flow to almost each of these surfaces is strictly ergodic, that is, it has a unique invariant normalized Borel measure (see Theorem 3.3 below). In particular, the above surfaces are ergodic components of the flow. We note that the larger the number of elements of $G(Q)$ the more uniform is the distribution of the surfaces $Q \times G(Q)v$ in the phase space and the closer is the billiard in Q to ergodicity. Proceeding from this observation and using approximation methods, it can be shown (see [2] and [3]) that there exists a massive (second-category) subset of the space of all polygons (and also of some its subspaces) formed by polygons with ergodic billiards. Namely, this is the subset of the polygons that can be sufficiently well approximated by rational ones. The arguments in [2] and [3] provide no information on how good this approximation must be.

In the present paper we construct a set with the same properties explicitly; namely, we indicate an order of approximation of a polygon by rational polygons ensuring the ergodicity of the billiard flow in it (Theorem 1.1).

Definition 1.3. Let $\delta(N)$ be a positive function of a positive integer variable that decreases to zero as $N \to \infty$. We say that a polygon Q admits an *approximation by rational polygons* at the rate $\delta(N)$ if there exist arbitrarily large numbers N such that the angles $\alpha_1, \alpha_2, \dots, \alpha_k$ between the adjacent sides of Q can be approximated with precision $\delta(N)$ by angles of the form $\pi\frac{n}{N}$, where n is an integer and the fractions $\frac{n_1}{N}, \frac{n_2}{N}, \dots, \frac{n_k}{N}$ corresponding to distinct angles cannot be cancelled by the same integer.

Theorem 1.1. *Let Q be a polygon admitting an approximation by rational polygons at the rate*

$$\delta(N) = \left(2^{2^{2^{N}}}\right)^{-1}.$$

Then the billiard flow in Q is ergodic.

The polygons satisfying the condition described in this theorem form a massive subset of the space of all polygons. A simple example of such a polygon is a right triangle with acute angle $\pi\,(a_5^{-1} + a_{10}^{-1} + \dots + a_{5n}^{-1} + \dots)$, where $\{a_n\}$ is a sequence defined by the relations $a_0 = 1$ and $a_{n+1} = 2^{a_n}$ for $n = 0, 1, 2, \dots$.

The scheme of the proof of Theorem 1.1 is as follows: first we carry out constructive estimates relating to the approximation of polygons and billiard flows in them (Proposition 2.3); this is the easier part of the proof, which we tackle in § 2. Then we prove a constructive version of the ergodic theorem for the billiard flow in a rational polygon (Theorem 3.1); this is the contents of § 3. Theorem 3.1 can be reduced to a similar result for the geodesic flow on a surface with flat structure (Theorem 3.2). It turns out that the proof of Theorem 3.2 requires a constructive version of a theorem of Masur [4] on the quadratic growth of the number of

saddle connections (singular
established in § 4. We present
Since there is little hope tha
of the Teichmüller theory use
use in this paper the ideas of
The idea of finding concre
by overcoming the non-const
theory in the arguments of K
The author is indebted to hi
research.
The results of this paper w

2. Appro

In this section, we carry ou
of polygons and billiard flows
formulated in § 1. Its proof i
rational polygons (Theorem 3

Definition 2.1. We call a pol
a homeomorphism $\varphi\colon Q \to$
the vertices of the two polygo
vertices are at most δ.

We denote by $d(Q)$ the sm
and diagonals of Q. If $\widetilde{Q}$ is a δ
Let $k(Q)$ be the number of sid
The following result justifies
duced earlier.

Lemma 2.1. *Assume that th*
$N > 0$ be approximated with
If $\delta < \frac{d(Q)}{2k(Q)D(Q)}$, then ther
the angles between its sides ar

Proof. Assume that the boun
$\dots$, A_{ik_i} be the consecutive
choose points $\widetilde{A}_{ij}$, $1 \leqslant i \leqslant l$, 1
(1) $\widetilde{A}_{i1} = A_{i1}$; moreover,
(2) the angle between the
between $\widetilde{A}_{i,j-1}\widetilde{A}_{ij}$ an
integer n, and their dif
or between $A_{i,j-1}A_{ij}$
(3) the lengths of the segm
By construction, the dista
$(j-1)D(Q)\delta < k(Q)D(Q)\delta$.

saddle connections (singular trajectories) in a flat structure, and this estimate is established in § 4. We present the proof of Theorem 1.1 proper at the end of § 2.

Since there is little hope that the arguments in [2] and, in particular, the methods of the Teichmüller theory used there, can be made more constructive, we mostly use in this paper the ideas of Boshernitzan [5] and Masur [4].

The idea of finding concrete examples of polygons with ergodic billiard flows by overcoming the non-constructivity stemming from the use of the Teichmüller theory in the arguments of Kerckhoff, Masur, and Smillie is due to A. M. Stëpin. The author is indebted to him for setting the problem and constant help in the research.

The results of this paper were announced in [6].

2. Approximation by rational polygons

In this section, we carry out quantitative estimates related to the approximation of polygons and billiard flows in them. After this we prove our central theorem formulated in § 1. Its proof is based on the ergodic properties of the billiards in rational polygons (Theorem 3.1) that we establish in § 3.

Definition 2.1. We call a polygon $\widetilde{Q}$ a *δ-perturbation* of a polygon Q if there exists a homeomorphism $\varphi\colon Q \to \widetilde{Q}$ establishing a one-to-one correspondence between the vertices of the two polygons such that the distances between the corresponding vertices are at most δ.

We denote by $d(Q)$ the smallest non-zero distance between the vertices, sides, and diagonals of Q. If $\widetilde{Q}$ is a δ-perturbation of Q, then, clearly, $|d(\widetilde{Q}) - d(Q)| \leqslant 2\delta$. Let $k(Q)$ be the number of sides of Q and let $D(Q)$ be its diameter.

The following result justifies the term 'approximation by rational polygons' introduced earlier.

Lemma 2.1. *Assume that the angles between the sides of a polygon Q can for some $N > 0$ be approximated with precision δ by angles of the form $\pi\frac{n}{N}$ with integer n. If $\delta < \dfrac{d(Q)}{2k(Q)D(Q)}$, then there exists a $k(Q)D(Q)\delta$-perturbation $\widetilde{Q}$ of Q such that the angles between its sides are precisely of the form $\pi\frac{n}{N}$ with integer n.*

Proof. Assume that the boundary of Q consists of l components and let $A_{i1}, A_{i2}, \dots, A_{ik_i}$ be the consecutive vertices of its ith component ($1 \leqslant i \leqslant l$). We now choose points $\widetilde{A}_{ij}$, $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant k_i$, with the following requirements in mind:

(1) $\widetilde{A}_{i1} = A_{i1}$; moreover, $\widetilde{A}_{12} = A_{12}$;

(2) the angle between the segments $\widetilde{A}_{i1}\widetilde{A}_{i2}$ and $\widetilde{A}_{11}\widetilde{A}_{12}$, and also the angles between $\widetilde{A}_{i,j-1}\widetilde{A}_{ij}$ and $\widetilde{A}_{ij}\widetilde{A}_{i,j+1}$ ($1 < j < k_i$) are of the form $\pi\frac{n}{N}$ with integer n, and their differences from the angles between $A_{i1}A_{i2}$ and $A_{11}A_{12}$ or between $A_{i,j-1}A_{ij}$ and $A_{ij}A_{i,j+1}$, respectively, are at most δ;

(3) the lengths of the segments $\widetilde{A}_{ij}\widetilde{A}_{i,j+1}$ and $A_{ij}A_{i,j+1}$ are the same.

By construction, the distance between $\widetilde{A}_{ij}$ and A_{ij} is not larger than $(j-1)D(Q)\delta < k(Q)D(Q)\delta$. Since $k(Q)D(Q)\delta < \frac{1}{2}\,d(Q)$, the $k(Q)$ line segments

$\widetilde{A}_{i1}\widetilde{A}_{i2}, \dots, \widetilde{A}_{i,k_i-1}\widetilde{A}_{ik_i}, \widetilde{A}_{ik_i}\widetilde{A}_{i1}$ $(1 \leqslant i \leqslant l)$ are pairwise disjoint, therefore they are the sides of some polygon $\widetilde{Q}$. This polygon is the required one.

Lemma 2.2. *Let $K > 0$ be an integer such that the polygon Q has angles not smaller than π/K. Then the sum of the lengths of K consecutive segments of an arbitrary billiard trajectory in Q is at least $d(Q)$.*

Proof. First we consider the billiard in a sector of angle α. We use the construction of straightening a billiard trajectory: as the trajectory reaches a side of the sector we reflect the sector with respect to this side and extend the trajectory into the reflected sector. As a result we obtain a linear trajectory passing successively through several copies of the original sector. This construction immediately shows that a billiard trajectory in a sector can have K finite segments only if $K\alpha < \pi$. Hence K consecutive segments of a billiard trajectory in the polygon Q cannot all have end-points on the sides of the same corner in this polygon and there exists either a segment with edges on some sides of Q with no common vertex or three consecutive vertices A, B, and C of the trajectory lying on three distinct sides a, b, and c of the polygon. In the first case the length of the corresponding segment is at least $d(Q)$. In the second case the sum of the lengths of AB and BC is not smaller than the distance between a and the segment $\widetilde{c}$ symmetric with c relative to the side b. It is easy to see that this distance is at least $d(Q)$.

Let Q be a polygon cut into triangles by diagonals in an arbitrary manner. We consider a polygon $\widetilde{Q}$ that is a δ-perturbation of Q. Assume that for each two vertices of Q that can be joined by a diagonal (lying inside Q) the corresponding vertices of $\widetilde{Q}$ can also be joined by a diagonal (at any rate, this holds for $\delta < \frac{1}{2}\,d(Q)$). Then there exists a partitioning of $\widetilde{Q}$ into triangles corresponding to the above partitioning of Q. Let φ be the homeomorphism of Q onto $\widetilde{Q}$ that is affine on each triangle in the triangulation of Q and maps it onto the corresponding triangle of $\widetilde{Q}$. Clearly, the distance between x and $\varphi(x)$ is at most δ for each $x \in Q$.

Let $\{T^t_Q\}$ and $\{T^t_{\widetilde{Q}}\}$ be the billiard flows in Q and $\widetilde{Q}$, respectively. For each $t \geqslant 0$ we define the functions x_t, v_t, $\widetilde{x}_t$, and $\widetilde{v}_t$ on $Q \times S^1$ by the following formulae: $(x_t(x,v), v_t(x,v)) = T^t_Q(x,v)$ and $(\widetilde{x}_t(x,v), \widetilde{v}_t(x,v)) = T^t_{\widetilde{Q}}(\varphi(x),v)$ for each (x,v) in $Q \times S^1$.

Proposition 2.3. *For each $t \geqslant 0$ there exists a set $B \subset Q \times S^1$ dependent on the polygons Q, $\widetilde{Q}$, the map φ and t, and of measure at most $C_3(C_1 t + C_2)^3\delta$ such that for each $(x,v) \in Q \times S^1$ outside B and for each τ, $0 \leqslant \tau \leqslant t$, at least one of the following two possibilities holds*:

1. *the distance between $x_t(x,v)$ and $\varphi^{-1}(\widetilde{x}_t(x,v))$ is at most $C_4(C_1 t + C_2)^2\delta$ and the angle between the directions of $v_t(x,v)$ and $\widetilde{v}_t(x,v)$ is at most $C_5(C_1 t + C_2)\delta$;*
2. *the points $x_t(x,v)$ and $\widetilde{x}_t(x,v)$ lie at a distance at most $C_6(C_1 t + C_2)^2\delta$ from the boundaries of Q and $\widetilde{Q}$, respectively.*

Here C_1, C_2, C_3, C_4, C_5, and C_6 are positive constants dependent on Q.

irwise disjoint, therefore they are required one.

at the polygon Q has angles not of K consecutive segments of an

angle α. We use the construction ctory reaches a side of the sector d extend the trajectory into the r trajectory passing successively construction immediately shows finite segments only if $K\alpha < \pi$. tory in the polygon Q cannot all in this polygon and there exists with no common vertex or three lying on three distinct sides a, b, f the corresponding segment is at ths of AB and BC is not smaller symmetric with c relative to the st $d(Q)$.

mals in an arbitrary manner. We f Q. Assume that for each two ying inside Q) the corresponding y rate, this holds for $\delta < \frac{1}{2} d(Q)$). gles corresponding to the above of Q onto $\widetilde{Q}$ that is affine on each the corresponding triangle of $\widetilde{Q}$. st δ for each $x \in Q$.

Q and $\widetilde{Q}$, respectively. For each $Q \times S^1$ by the following formulae: $v)) = T^t_{\widetilde{Q}}(\varphi(x), v)$ for each (x, v)

set $B \subset Q \times S^1$ dependent on the at most $C_3(C_1 t + C_2)^3\delta$ such that τ, $0 \leqslant \tau \leqslant t$, at least one of the

$x, v)$) is at most $C_4(C_1 t + C_2)^2\delta$ $v_t(x, v)$ and $\widetilde{v}_t(x, v)$ is at most

listance at most $C_6(C_1 t + C_2)^2\delta$ ely.

nstants dependent on Q.

Proof. We index the sides of Q and $\widetilde{Q}$ by the numbers $1, 2, \dots, k(Q)$ in a coordinated manner. Let α_i be the angle between their sides with index i. Since Q and $\widetilde{Q}$ are δ-perturbations of each other, it follows that $\sin\alpha_i \leqslant 2\delta/l_i$, where l_i is the length of the ith side of Q. Hence $\alpha_i \leqslant \pi/2 \cdot 2\delta/l_i \leqslant \pi\delta/d(Q)$.

Let $i_1, i_2, \dots, i_n$ be an arbitrary sequence of indices, $1 \leqslant i_j \leqslant k(Q)$. Let Q_1 be the polygon symmetric to Q with respect to the i_1th side, let Q_2 be the polygon symmetric to Q_1 with respect to the i_2th side (the indexing of the sides of Q_1 is carried over from Q), and so on. In a similar way, starting from $\widetilde{Q}$ we construct the sequence of reflected polygons $\widetilde{Q}_1, \widetilde{Q}_2, \dots, \widetilde{Q}_n$. We also set $Q_0 = Q$ and $\widetilde{Q}_0 = \widetilde{Q}$. Let R_i and $\widetilde{R}_i$ be the reflections of Q and $\widetilde{Q}$, respectively, with respect to their ith sides. Then the plane motion taking Q to Q_j (with the indexing of sides taken into account) is obviously $R_{i_1} R_{i_2} \cdots R_{i_j}$. In the same way the map $\widetilde{R}_{i_1} \widetilde{R}_{i_2} \cdots \widetilde{R}_{i_j}$ takes $\widetilde{Q}$ into $\widetilde{Q}_j$.

In what follows, for an arbitrary pair of polygons Q' and Q'' that are isometric images of Q and $\widetilde{Q}$, we denote the largest angle between the corresponding sides of Q' and Q'' by $\beta(Q', Q'')$, and the largest distance between the corresponding vertices of these polygons by $\rho(Q', Q'')$.

We shall now prove by induction on j that

$$\beta(Q_j, \widetilde{Q}_j) \leqslant \pi \frac{\delta}{d(Q)}(4j + 1) \quad \text{and} \quad \rho(Q_j, \widetilde{Q}_j) \leqslant 2\pi \frac{D(Q)}{d(Q)}(j+1)^2\delta$$

for $0 \leqslant j \leqslant n$. For $j = 0$ these estimates are consequences of those already obtained. Now assume that we have proved this assertion for some value of j, $0 \leqslant j < n$. Let Q'_j and Q'_{j+1} be the images of Q under the isometries $\widetilde{R}_{i_1} \widetilde{R}_{i_2} \cdots \widetilde{R}_{i_j}$ and $\widetilde{R}_{i_1} \widetilde{R}_{i_2} \cdots \widetilde{R}_{i_{j+1}}$, respectively, and let Q''_{j+1} be the polygon symmetric with Q'_j relative to the side with index i_{j+1}. Then

$$\rho(Q'_j, \widetilde{Q}_j) = \rho(Q'_{j+1}, \widetilde{Q}_{j+1}) = \rho(Q, \widetilde{Q}) \leqslant \delta,$$

$$\beta(Q'_j, \widetilde{Q}_j) = \beta(Q'_{j+1}, \widetilde{Q}_{j+1}) = \beta(Q, \widetilde{Q}) \leqslant \pi \frac{\delta}{d(Q)}.$$

Let p' be the vertex of Q'_{j+1} that is the end-point of the i_{j+1}th side and let p'' and p be the corresponding vertices of Q''_{j+1} and Q_{j+1}. Then the polygon Q''_{j+1} can be obtained from Q'_{j+1} by means of the translation that takes p' into p'' and a subsequent rotation with centre at p''. The points p' and p'' are at a distance not greater than δ from the corresponding vertices of $\widetilde{Q}_{j+1}$ and $\widetilde{Q}_j$, therefore the distance between them is at most 2δ. The angle of the rotation of Q'_{j+1} after the translation is double the angle between the i_{j+1}th sides of Q'_j and $\widetilde{Q}_j$, therefore it is $2\pi \dfrac{\delta}{d(Q)}$ at most. Hence

$$\beta(Q''_{j+1}, \widetilde{Q}_{j+1}) \leqslant \beta(Q''_{j+1}, Q'_{j+1}) + \beta(Q'_{j+1}, \widetilde{Q}_{j+1}) \leqslant 2\pi \frac{\delta}{d(Q)} + \pi \frac{\delta}{d(Q)} = 3\pi \frac{\delta}{d(Q)},$$

$$\rho(Q''_{j+1}, \widetilde{Q}_{j+1}) \leqslant \rho(Q''_{j+1}, Q'_{j+1}) + \rho(Q'_{j+1}, \widetilde{Q}_{j+1}) \leqslant \left(2\delta + D(Q) \cdot 2\pi \frac{\delta}{d(Q)}\right) + \delta.$$

Next, by the induction hypothesis

$$\beta(Q_j,\widetilde{Q}_j) \leqslant \pi\,\frac{\delta}{d(Q)}(4j+1) \quad\text{and}\quad \rho(Q_j,\widetilde{Q}_j) \leqslant 2\pi\,\frac{D(Q)}{d(Q)}(j+1)^2\delta.$$

Consequently,

$$\beta(Q_j,Q_j') \leqslant \pi\,\frac{\delta}{d(Q)}(4j+2) \quad\text{and}\quad \rho(Q_j,Q_j') \leqslant 2\pi\,\frac{D(Q)}{d(Q)}(j+1)^2\delta+\delta.$$

Since $\beta(Q_j,Q_j') = \beta(Q_{j+1},Q_{j+1}'')$, it follows that

$$\beta(Q_{j+1},\widetilde{Q}_{j+1}) \leqslant \beta(Q_{j+1},Q_{j+1}'') + \beta(Q_{j+1}'',\widetilde{Q}_{j+1}) \leqslant \pi\,\frac{\delta}{d(Q)}\bigl(4(j+1)+1\bigr).$$

We can obtain Q_{j+1} from Q_{j+1}'' by making the translation that sends p'' into p and then rotating around p. The distance between p'' and p is not greater than $\rho(Q_j,Q_j')$, while the rotation angle is not larger than $\beta(Q_j,Q_j')$, therefore we have the inequality $\rho(Q_{j+1},Q_{j+1}'') \leqslant \rho(Q_j,Q_j') + D(Q)\cdot\beta(Q_j,Q_j')$. Hence

$$\begin{aligned}\rho(Q_{j+1},\widetilde{Q}_{j+1}) &\leqslant \rho(Q_j,Q_j') + D(Q)\cdot\beta(Q_j,Q_j') + \rho(Q_{j+1}'',\widetilde{Q}_{j+1})\\ &\leqslant \left(2\pi\,\frac{D(Q)}{d(Q)}(j+1)^2\delta+\delta\right) + \pi\,\frac{D(Q)}{d(Q)}(4j+2)\delta + \left(3\delta + 2\pi\,\frac{D(Q)}{d(Q)}\delta\right)\\ &\leqslant 2\pi\,\frac{D(Q)}{d(Q)}(j+2)^2\delta.\end{aligned}$$

This completes the proof of the inductive step.

We now fix an arbitrary $t \geqslant 0$. Let $(x,v) \in Q\times S^1$ and let t_1 be some instant of time, $0 \leqslant t_1 \leqslant t + D(Q)$. We consider the case when, by the time t_1, the billiard trajectories $(x_\tau(x,v),v_\tau(x,v))$ and $(\widetilde{x}_\tau(x,v),\widetilde{v}_\tau(x,v))$ have been driven back equally often from the boundaries of Q and $\widetilde{Q}$, respectively; moreover, we assume that the sides involved have had the same indices $i_1,i_2,\dots,i_n$ in both cases. We now use the straightening of the trajectories in question, which we already used in the proof of Lemma 2.2. As a result we obtain two linear trajectories, X_τ and $\widetilde{X}_\tau$, starting from x and $\varphi(x)$ in the direction v and passing consecutively through the polygons $Q_0 = Q, Q_1,\dots,Q_n$ or $\widetilde{Q}_0 = \widetilde{Q},\widetilde{Q}_1,\dots,\widetilde{Q}_n$, respectively, where $Q_j = R_{i_1}R_{i_2}\cdots R_{i_j}(Q)$ and $\widetilde{Q}_j = \widetilde{R}_{i_1}\widetilde{R}_{i_2}\cdots\widetilde{R}_{i_j}(\widetilde{Q})$, $1\leqslant j\leqslant n$. The distance between X_τ and $\widetilde{X}_\tau$ is equal for each τ to the distance between x and $\varphi(x)$. As follows from the definition of φ, the latter is at most δ. Let Q' be the image of Q_n under the map $(\widetilde{R}_{i_1}\widetilde{R}_{i_2}\cdots\widetilde{R}_{i_n})^{-1}$ and let x' be the image of X_{t_1} under the same map. Then the distance between x' and $\widetilde{x}_{t_1}$ is equal to that between X_{t_1} and $\widetilde{X}_{t_1}$, therefore it is at most δ. The transformation $R = (R_{i_1}R_{i_2}\cdots R_{i_n})^{-1}\widetilde{R}_{i_1}\widetilde{R}_{i_2}\cdots\widetilde{R}_{i_n}$ takes Q' to Q and the point x' of Q' is taken to x_{t_1}. Since R is an isometry, the distance between x' and x_{t_1} is at most $\rho(Q',Q)$. Hence the distance between x_{t_1} and $\widetilde{x}_{t_1}$ is not greater than

$$\delta + \rho(Q',Q) \leqslant \delta + \rho(Q',\widetilde{Q}) + \rho(\widetilde{Q},Q) \leqslant 2\delta + \rho(Q',\widetilde{Q}).$$

The quantity $\rho(Q', \widetilde{Q})$ is equal to $\rho(Q_n, \widetilde{Q}_n)$, while the latter quantity does not exceed $2\pi \dfrac{D(Q)}{d(Q)}(n+1)^2\delta$, as has already been proved. Hence the distance between x_{t_1} and $\widetilde{x}_{t_1}$ is at most

$$2\delta + 2\pi \frac{D(Q)}{d(Q)} (n+1)^2\delta \leqslant C(N_t+1)^2\delta,$$

where $C = 3\pi \dfrac{D(Q)}{d(Q)}$ and N_t is the largest number of rebounds of a billiard trajectory in Q by the time $t + D(Q)$. Incidentally, the distance between x_{t_1} and $\varphi^{-1}(\widetilde{x}_{t_1})$ is at most $(C+1)(N_t+1)^2\delta$.

As regards the directions v_{t_1} and $\widetilde{v}_{t_1}$, they are mapped to v by the transformations $R_{i_1}R_{i_2}\cdots R_{i_n}$ and $\widetilde{R}_{i_1}\widetilde{R}_{i_2}\cdots\widetilde{R}_{i_n}$, respectively, therefore the angle between them is at most $2(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n})$, which on its part is less than or equal to $2n \cdot \pi \dfrac{\delta}{d(Q)} \leqslant \dfrac{2\pi}{d(Q)} N_t \delta$.

We now consider the case when one of the trajectories (x_τ, v_τ) and $(\widetilde{x}_\tau, \widetilde{v}_\tau)$ has made n rebounds at the boundary of the corresponding polygon by the time t_1, $0 \leqslant t_1 \leqslant t$, while the other has made one rebound fewer. Moreover, we assume that both trajectories will have made n rebounds some time later and all these n rebounds will have occurred at sides with equal indices. Let t_- and t_+ ($t_- \leqslant t_+$) be the times of the nth rebounds and let i be the index of the corresponding sides of Q and $\widetilde{Q}$. Clearly, $0 \leqslant t_- \leqslant t_1 \leqslant t_+ \leqslant t + D(Q)$. It follows from the above that the distances between x_τ and $\widetilde{x}_\tau$ at times t_- and t_+ are not greater than $C(N_t+1)^2\delta$. At each of these instances, one of the points is on the ith side of the corresponding polygon while the other lies at a distance not greater than

$$C(N_t+1)^2\delta + \rho(Q, \widetilde{Q}) \leqslant C(N_t+1)^2\delta + \delta \leqslant (C+1)(N_t+1)^2\delta$$

from the ith side of the corresponding polygon. Since the trajectories (x_τ, v_τ) and $(\widetilde{x}_\tau, \widetilde{v}_\tau)$ do not hit the boundaries in the period of time between t_- and t_+, for each instant in $[t_-, t_+]$ and, in particular, for $\tau = t_1$ the distance from x_τ or $\widetilde{x}_\tau$ to the boundary of Q or $\widetilde{Q}$, respectively, is at most $(C+1)(N_t+1)^2\delta$.

We now partition $Q \times S^1$ into two subsets $A(t)$ and $B(t)$. We assign a point $(x, v) \in Q \times S^1$ to $A(t)$ if there exists $t' \geqslant t$ such that, by the time t', the trajectories $(x_\tau(x,v), v_\tau(x,v))$ and $(\widetilde{x}_\tau(x,v), \widetilde{v}_\tau(x,v))$ have hit the boundaries equally often and at sides of Q and $\widetilde{Q}$ with the same indices, and if, moreover, the difference between the numbers of hits at the boundary made by these trajectories by an arbitrary time τ, $0 \leqslant \tau \leqslant t'$, is at most 1. The set $B(t)$ consists of those elements of $Q \times S^1$ left outside $A(t)$.

By Lemma 2.2 the quantity $N_t + 1$ is not larger than $C_1 t + C_2$, where C_1 and C_2 are certain constants dependent on Q. We set $C_4 = C_6 = C + 1$ and $C_5 = 2\pi/d(Q)$. Then we obtain in view of the above that for each $(x, v) \in A(t)$ and for each τ, $0 \leqslant \tau \leqslant t$, one of the properties (1) and (2) in the statement of the proposition must be satisfied.

For each $(x, v) \in B(t)$ there exists t_1, $0 \leqslant t_1 \leqslant t$, such that the billiard trajectories $(x_\tau(x,v), v_\tau(x,v))$ and $(\widetilde{x}_\tau(x,v), \widetilde{v}_\tau(x,v))$ have hit the boundary equally often by this time and at sides of Q and $\widetilde{Q}$ with the same indices, but afterwards, either (a) one of the trajectories goes directly to a vertex of the corresponding polygon, or (b) the next sides hit by these trajectories have distinct indices, or (c) one of the trajectories hits the boundary twice before another hits it once. Let $B_0(t, C')$ be the set of $(y, u) \in Q \times S^1$, such that the straight line through y in the direction u passes at a distance less than $C'(C_1 t + C_2)^2 \delta$ from some vertex of Q. It is easy to show that in each of the above cases (a), (b), and (c) the pair (x_{t_1}, v_{t_1}) belongs to $B_0(t, C')$ for sufficiently large values of C' dependent on Q (but not on t). Hence the set $B_1(t, C')$ of elements $(x, v) \in Q \times S^1$ occurring in $B_0(t, C')$ under the action of the billiard flow in Q at time t or earlier contains $B(t)$ for the same value of C'.

For an arbitrary direction $v \in S^1$ the measure of the set of $x \in Q$ such that $(x, v) \in B_0(t, C')$ clearly has the upper estimate $2C'(C_1 t + C_2)^2 \delta \cdot D(Q)\, k(Q)$. Hence

$$\mu \times \lambda\big(B_0(t, C')\big) \leqslant 2\pi \cdot 2C'(C_1 t + C_2)^2 \delta \cdot D(Q)\, k(Q).$$

Straightening the billiard trajectory we can obtain the same estimate for the measure of the set of elements $(x, v) \in Q \times S^1$ getting into $B_0(t, C')$ under the action of the billiard flow upon hitting some fixed sequence of sides. Let K be a number satisfying the condition described in Lemma 2.2. Then the elements $(x, v) \in Q \times S^1$ taken into $B_0(t, C')$ by the billiard flow before the time $d(Q)$ hit the sides of Q fewer than K times. In view of the above, the measure of the set of such elements is at most $(k(Q)+1)^K\, 4\pi C'(C_1 t + C_2)^2 \delta \cdot D(Q)\, k(Q)$. Since the measure $\mu \times \lambda$ is invariant with respect to the billiard flow, it follows that

$$\mu \times \lambda\big(B_1(t, C')\big) \leqslant \big(t/d(Q) + 1\big)\, \big(k(Q)+1\big)^K\, 4\pi C'(C_1 t + C_2)^2 \delta \cdot D(Q)\, k(Q),$$

which is not larger than $C_3\,(C_1 t + C_2)^3 \delta$ with some constant C_3 dependent on Q. Since the set $B(t)$ lies in $B_1(t, C')$, this completes the proof of the proposition.

Proof of Theorem 1.1. By the statistical ergodic theorem, to prove the ergodicity of the billiard flow in Q it suffices to establish that for each function $F(x, v)$ on $Q \times S^1$ that is integrable with respect to $\mu \times \lambda$ we have

$$\frac{1}{t} \int_0^t F\big(T_Q^\tau(x, v)\big)\, d\tau \to \frac{1}{2\pi \cdot \mathrm{Area}(Q)} \int_{Q \times S^1} F\, d(\mu \times \lambda) \quad \text{as } t \to \infty$$

in the space $L_1(Q \times S^1, \mu \times \lambda)$. It suffices to prove this convergence for a family of functions such that their linear combinations are dense in $L_1(Q \times S^1, \mu \times \lambda)$. Let this be the family of functions of the form $F(x, v) = f(x)\, h(v)$, where f is a Lipschitz function in Q vanishing at the boundary of Q and h is a Lipschitz function on S^1. Further, since $\mu \times \lambda$ is invariant with respect to $\{T_Q^t\}$, the function

$$q(t) = \int_{Q \times S^1} \left| \frac{1}{t} \int_0^t F\big(T_Q^\tau(x, v)\big)\, d\tau - \frac{1}{2\pi \cdot \mathrm{Area}(Q)} \int_{Q \times S^1} F\, d(\mu \times \lambda) \right| d\mu(x)\, d\lambda(v)$$

satisfies the relation $(t + t')\, q(t + t') \leqslant t\, q(t) + t'\, q(t')$ $(t, t' > 0)$. Consequently, we have $\lim_{t \to \infty} q(t) = \inf_{t > 0} q(t)$ and it suffices to show that $q(t)$ takes arbitrarily small values.

Let $N > 0$ be an integer such that the angles between the sides of Q can be approximated with precision $\delta(N)$ by angles of the form $\pi \frac{n}{N}$, where n is an integer and the fractions $\frac{n_1}{N}, \frac{n_2}{N}, \dots, \frac{n_k}{N}$ corresponding to distinct angles cannot be cancelled by the same integer. By hypothesis, we can choose N arbitrarily large.

We set $\Delta = k(Q)D(Q)\delta(N)$. For sufficiently large N the value of Δ is smaller than $\frac{1}{2}\, d(Q)$. By Lemma 2.1 there exists a Δ-perturbation $\widetilde{Q}$ of Q such that the angles between its sides are of the form $\pi \frac{n}{N}$ with integer n. The angles between the pairs of corresponding sides of Q and $\widetilde{Q}$ are $\pi \cdot \Delta/d(Q)$-close (see the proof of Proposition 2.3). For N sufficiently large this quantity is smaller than π/N, in which case the angles between the sides of $\widetilde{Q}$ are unambiguously defined, that is, they are equal to the angles $\pi \frac{n_1}{N}, \pi \frac{n_2}{N}, \dots, \pi \frac{n_k}{N}$ indicated above. We note that rotations through angles twice as big generate the rotation subgroup of $G(\widetilde{Q})$. Since the fractions $\frac{n_1}{N}, \frac{n_2}{N}, \dots, \frac{n_k}{N}$ are simultaneously uncancellable, this subgroup is cyclic and generated by the rotation through $2\pi/N$, while the order $r(\widetilde{Q})$ of $G(\widetilde{Q})$ is $2N$.

We now consider an arbitrary triangulation of Q by diagonals, the analogous triangulation of $\widetilde{Q}$ (this is well defined since $\Delta < \frac{1}{2}\, d(Q)$), and define a map φ of Q onto $\widetilde{Q}$ taking each triangle in Q affinely to the corresponding triangle in $\widetilde{Q}$. The distance between x and $\varphi(x)$ is at most Δ for each $x \in Q$. Let P be a triangle in the partitioning of Q and let φ_P be the linear part of the restriction of φ to P. For each vector v we can find a segment of length $d(Q)$ parallel to v in P. Hence $|\varphi_P(v) - v| \leqslant 2\Delta/d(Q)|v|$. From now on, we assume that N is sufficiently large so that $\Delta < \frac{1}{10}\, d(Q)$. Then $|\varphi_P(v)| \geqslant \frac{4}{5}\, |v|$, that is, the map φ^{-1} is Lipschitz on the triangle $\varphi(P)$, with Lipschitz constant $\frac{5}{4}$. Since P was chosen arbitrarily, φ^{-1} is Lipschitz on the entire polygon $\widetilde{Q}$. We now estimate the determinant of φ_P, that is, the Jacobian $J(\varphi)$ of φ. The entries of the matrix corresponding to φ_P with respect to an orthonormal basis are $2\Delta/d(Q)$-close to the entries of the identity matrix. Hence

$$|J(\varphi) - 1| \leqslant \left(1 + \frac{2\Delta}{d(Q)}\right)^2 + \left(\frac{2\Delta}{d(Q)}\right)^2 - 1 = 4\,\frac{\Delta}{d(Q)}\left(\frac{\Delta}{d(Q)} + 1\right) < 5\,\frac{\Delta}{d(Q)}\,.$$

In particular, $\frac{1}{2} \leqslant J(\varphi) \leqslant \frac{3}{2}$.

We now define a function $\widetilde{F}$ on $\widetilde{Q} \times S^1$ as follows: $\widetilde{F}(x,v) = F(\varphi^{-1}(x), v)$ for each element $(x,v) \in \widetilde{Q} \times S^1$. We recall that $F(x,v)$ has the form $f(x)\,h(v)$, where f and h are Lipschitz functions on Q and S^1, respectively, and f vanishes at the boundary of Q. Let $L_0, E_0 > 0$ be such that L_0 is the Lipschitz constant for f and h, and assume that $|f(x)|$ and $|h(v)|$ are not larger than E_0 for all $x \in Q$, $v \in S^1$. Then, in view of the above, $f(\varphi^{-1}(x))$ is a Lipschitz function on $\widetilde{Q}$ with Lipschitz constant $\frac{5}{4}\, L_0$. Hence $F(x,v)$ and $\widetilde{F}(x,v)$ are not larger than $E = E_0^2$ in absolute value, are Lipschitz with respect to both variables with Lipschitz constant $L = \frac{5}{4} L_0 E_0$, and vanish for x at the boundary of Q or $\widetilde{Q}$, respectively.

We now introduce the constants $C' = 4\ \mathrm{Area}(Q) \cdot L/E$ and $C'' = \alpha/\pi$, where α is the sum of all the interior angles of Q, and $C''' = 16\ \mathrm{Area}(Q)/d^2(Q)$. Further, we set $t_N = C'N \cdot N^{(H(C''N)\cdot C'''N^2)^{C''N+5}}$, where $H(n) = (500n)^{(2n)^{2n}}$.

The quantity $q(t_N)$ has the following estimate:

$$q(t_N) \leqslant q_1(t_N) + q_2(t_N) + q_3 + q_4,$$

where

$$q_1(t) = \int_{Q\times S^1} \left| \frac{1}{t}\int_0^t F\big(T_Q^\tau(x,v)\big)\,d\tau - \frac{1}{t}\int_0^t \widetilde{F}\big(T_{\widetilde{Q}}^\tau(\varphi(x),v)\big)\,d\tau \right| d\mu(x)\,d\lambda(v),$$

$$q_2(t) = \int_{Q\times S^1} \left| \frac{1}{t}\int_0^t \widetilde{F}\big(T_{\widetilde{Q}}^\tau(\varphi(x),v)\big)\,d\tau - \frac{1}{r(\widetilde{Q})\ \mathrm{Area}(\widetilde{Q})} \sum_{g\in G(\widetilde{Q})} \int_{\widetilde{Q}} \widetilde{F}(y,gv)\,d\mu(y) \right| d\mu(x)\,d\lambda(v),$$

$$q_3 = \frac{\mathrm{Area}(Q)}{\mathrm{Area}(\widetilde{Q})} \int_{S^1} \left| \frac{1}{r(\widetilde{Q})} \sum_{g\in G(\widetilde{Q})} \int_{\widetilde{Q}} \widetilde{F}(y,gv)\,d\mu(y) - \frac{1}{2\pi}\int_{\widetilde{Q}\times S^1} \widetilde{F}\,d(\mu\times\lambda) \right| d\lambda(v),$$

$$q_4 = \mathrm{Area}(Q) \left| \frac{1}{\mathrm{Area}(\widetilde{Q})} \int_{\widetilde{Q}\times S^1} \widetilde{F}\,d(\mu\times\lambda) - \frac{1}{\mathrm{Area}(Q)} \int_{Q\times S^1} F\,d(\mu\times\lambda) \right|.$$

We now verify that each of the terms in the above sum is arbitrarily small for N large. The estimate of q_4 is the easiest, for

$$q_4 \leqslant \left| \frac{\mathrm{Area}(Q)}{\mathrm{Area}(\widetilde{Q})} - 1 \right| \cdot E + \left| \int_{\widetilde{Q}\times S^1} \widetilde{F}\,d(\mu\times\lambda) - \int_{Q\times S^1} F\,d(\mu\times\lambda) \right|.$$

However,

$$\mathrm{Area}(\widetilde{Q}) = \int_Q J(\varphi)\,d\mu, \qquad \int_{\widetilde{Q}\times S^1} \widetilde{F}\,d(\mu\times\lambda) = \int_{Q\times S^1} F(x,v)\,J(\varphi)(x)\,d\mu(x)\,d\lambda(v).$$

Hence, first,

$$\left| \int_{\widetilde{Q}\times S^1} \widetilde{F}\,d(\mu\times\lambda) - \int_{Q\times S^1} F\,d(\mu\times\lambda) \right| \leqslant 5E\,\frac{\Delta}{d(Q)} \cdot 2\pi\ \mathrm{Area}(Q),$$

and, moreover, $|\,\mathrm{Area}(\widetilde{Q}) - \mathrm{Area}(Q)| \leqslant 5\dfrac{\Delta}{d(Q)} \cdot \mathrm{Area}(Q)$; in particular, we see that $\frac{1}{2}\,\mathrm{Area}(Q) \leqslant \mathrm{Area}(\widetilde{Q}) \leqslant 2\,\mathrm{Area}(Q)$. Consequently

$$q_4 \leqslant 2\cdot 5\frac{\Delta}{d(Q)} \cdot E + 5E\frac{\Delta}{d(Q)} \cdot 2\pi\,\mathrm{Area}(Q),$$

which is small for large N.

We now estimate q_3 using the fact that the group $G(\widetilde{Q})$ contains a rotation through the angle $2\pi/N$. Hence for arbitrary directions $v, v_0 \in S^1$ we can find an element g_0 of $G(\widetilde{Q})$ such that the angle between v and $g_0 v_0$ is at most $2\pi/N$. In view of the properties of $\widetilde{F}$ we have $|\widetilde{F}(y, gv) - \widetilde{F}(y, gg_0v_0)| \leqslant 2\pi\, L/N$ for all $g \in G(\widetilde{Q})$ and $y \in \widetilde{Q}$. Adding these inequalities for all $g \in G(\widetilde{Q})$ and then integrating with respect to y, we arrive at the estimate

$$\left| \sum_{g\in G(\widetilde{Q})} \int_{\widetilde{Q}} \widetilde{F}(y, gv)\, d\mu(y) - \sum_{g\in G(\widetilde{Q})} \int_{\widetilde{Q}} \widetilde{F}(y, gv_0)\, d\mu(y) \right| \leqslant 2\pi \frac{L}{N} \cdot r(\widetilde{Q}) \operatorname{Area}(\widetilde{Q}).$$

On integrating also with respect to v_0 we obtain that $q_3 \leqslant 2\pi\, L/N \cdot \operatorname{Area}(Q)$.

Next, we pass in the integral $q_2(t_N)$ from the variables x, v to $\varphi(x), v$. In view of the inequality $J(\varphi^{-1}) = J(\varphi)^{-1} \leqslant 2$ we obtain

$$q_2(t) \leqslant 2 \int_{\widetilde{Q}\times S^1} \left| \frac{1}{t} \int_0^t \widetilde{F}\big(T^\tau_{\widetilde{Q}}(x, v)\big)\, d\tau - \frac{1}{r(\widetilde{Q}) \operatorname{Area}(\widetilde{Q})} \sum_{g\in G(\widetilde{Q})} \int_{\widetilde{Q}} \widetilde{F}(y, gv)\, d\mu(y) \right| d\mu(x)\, d\lambda(v).$$

We set $\varepsilon = 1/N$ in Theorem 3.1 (see § 3) to obtain

$$q_2(t_N) \leqslant 2 \cdot 8\pi E/N \cdot \operatorname{Area}(\widetilde{Q}) \leqslant 32\, E/N \cdot \operatorname{Area}(Q),$$

provided that

$$t_N \geqslant \frac{L \cdot S(\widetilde{Q})}{E} N^{\left(H(m)\cdot S(\widetilde{Q})/d^2(\widetilde{Q})\cdot N\right)^{m+5}},$$

where $S(\widetilde{Q}) = r(\widetilde{Q}) \operatorname{Area}(\widetilde{Q})$, $m = r(\widetilde{Q}) \cdot \widetilde{\alpha}/(2\pi)$, and $\widetilde{\alpha}$ is the sum of all the interior angles of the polygon $\widetilde{Q}$. This condition is satisfied because $r(\widetilde{Q}) = 2N$, $S(\widetilde{Q}) = 2N \operatorname{Area}(\widetilde{Q}) \leqslant 4N \operatorname{Area}(Q)$, $d(\widetilde{Q}) \geqslant d(Q)/2$, $\widetilde{\alpha} = \alpha$, and, consequently, $L \cdot S(\widetilde{Q})/E \leqslant C'N$, $m = C''N$, and $S(\widetilde{Q})/d^2(\widetilde{Q}) \leqslant C'''N$.

Finally, we obtain an estimate of $q_1(t_N)$ using Proposition 2.3. Let $B \subset Q \times S^1$ be the set associated by this proposition with the polygons Q and $\widetilde{Q}$, the map φ, and the time t_N. The measure of B is at most $C_3(C_1 t_N + C_2)^3\, \Delta$, where C_1, C_2, and C_3 are constants depending on Q. We consider an arbitrary point (x, v) lying in $Q \times S^1$ and outside B. For each τ, $0 \leqslant \tau \leqslant t_N$, we set $(x_\tau, v_\tau) = T^\tau_Q(x, v)$ and $(\widetilde{x}_\tau, \widetilde{v}_\tau) = T^\tau_{\widetilde{Q}}(\varphi(x), v)$. By Proposition 2.3 either the distance between x_τ and $\varphi^{-1}(\widetilde{x}_\tau)$ is not greater than $C_4(C_1 t_N + C_2)^2\Delta$ and the angle between v_τ and $\widetilde{v}_\tau$ is at most $C_5(C_1 t_N + C_2)\Delta$, or the points x_τ and $\widetilde{x}_\tau$ lie at distances not greater than $C_6(C_1 t_N + C_2)^2\Delta$ from the boundaries of Q and $\widetilde{Q}$, respectively (here C_4, C_5, and C_6 are constants dependent on Q). In the first case

$$|F(x_\tau, v_\tau) - \widetilde{F}(\widetilde{x}_\tau, \widetilde{v}_\tau)| \leqslant L \cdot \big(C_4(C_1 t_N + C_2)^2 + C_5(C_1 t_N + C_2)\big)\, \Delta,$$

while in the second case $|F(x_\tau, v_\tau)|$, $|\widetilde{F}(\widetilde{x}_\tau, \widetilde{v}_\tau)| \leqslant L \cdot C_6(C_1 t_N + C_2)^2\,\Delta$. Since τ was chosen arbitrarily, we see that the integrand in $q_1(t_N)$ is at most $L \cdot C\,(C_1 t_N + C_2)^2\,\Delta$ for the values of x and v in question, where C is a constant depending on Q. Hence

$$q_1(t_N) \leqslant L \cdot C(C_1 t_N + C_2)^2 \Delta \cdot 2\pi \operatorname{Area}(Q) + 2E \cdot C_3 (C_1 t_N + C_2)^3 \Delta.$$

We choose the function $\delta(N)$ so that the right-hand side of this inequality converges to zero as $N \to \infty$.

Hence $q(t)$ takes arbitrarily small values, as required.

3. Billiard in a rational polygon

In this section we prove results on the ergodic properties of the billiard in a rational polygon and of the geodesic flow on a surface with flat structure.

Let Q be a rational polygon of arbitrary form, let $G(Q)$ be the group generated by the linear parts of the reflections with respect to its sides, let $r(Q)$ be the order of $G(Q)$, let $\alpha(Q)$ be the sum of all interior angles of Q, and $s(Q)$ the length of its shortest generalized diagonal (that is, a billiard trajectory with end-points at vertices of Q). We also set $m(Q) = \dfrac{r(Q) \cdot \alpha(Q)}{2\pi}$ ($m(Q)$ is an integer) and $S(Q) = r(Q) \cdot \operatorname{Area}(Q)$.

We denote by $\{T^\tau\}$ the billiard flow in Q. For each measurable function $F(x,v)$ on the phase space $Q \times S^1$ of the billiard we denote by $S^t F(x,v)$ the average value of this function under the action of the flow $\{T^\tau\}$ over the period t, that is,

$$S^t F(x,v) = \frac{1}{t} \int_0^t F\bigl(T^\tau(x,v)\bigr)\, d\tau.$$

Theorem 3.1. *Assume that $L, E > 0$ and that $f_v(x) = F(x,v)$ is a Lipschitz function on Q with Lipschitz constant L for each direction $v \in S^1$; assume, moreover, that $F(x,v) = 0$ for $x \in \partial Q$ and $|F(x,v)| \leqslant E$ for all $x \in Q$ and $v \in S^1$. Then*

$$\frac{1}{\operatorname{Area}(Q)} \int_{Q \times S^1} \left| S^t F(x,v) - \frac{1}{S(Q)} \sum_{g \in G(Q)} \int_Q F(y, gv)\, d\mu(y) \right| d\mu(x)\, d\lambda(v) \leqslant 8\pi E \cdot \varepsilon$$

$$\text{for } t \geqslant \frac{L \cdot S(Q)}{E} \left(\frac{1}{\varepsilon}\right)^{\left(H(m(Q)) \cdot S(Q)/s^2(Q) \cdot 1/\varepsilon\right)^{m(Q)+5}}$$

and for each ε, $0 < \varepsilon \leqslant 0.999$, where $H(1) = 2^{60}$, $H(m) = (500m)^{(2m)^{2m}}$ for $m > 1$, μ is Lebesgue measure on Q, and λ is Lebesgue measure on S^1 normalized so that $\lambda(S^1) = 2\pi$.

We shall reduce this theorem to Theorem 3.2 concerning flat structures.

Definition 3.1. A *flat structure* on a compact connected oriented surface M is an atlas $\omega = \{(U_\alpha, f_\alpha)\}$ of charts (where U_α is a subdomain of M and f_α is a homeomorphism of U_α onto a subdomain of $\mathbb{R}^2$) such that

- all the transition functions are translations of $\mathbb{R}^2$;
- the domains U_α cover the whole of M except for finitely many points, which are said to be *singular*;

- a punctured neighbou
of a punctured neighb
is a translation with
is called the *multiplic*

Equivalently, a flat struct
many singular points such t
singular point, where m is a
structure ω gives rise to a g
everywhere (if a trajectory a
any further), but only on a
measure μ_ω associated with
that is, the phase space $M \times$
the invariant surfaces $M \times$
invariant surface as a flow o

Definition 3.2. A *saddle*
joining two singular points (
interior points.

Let ω be an arbitrary flat
to the measure μ_ω, let s b
let m be the sum of the m
what follows we assume tha
declare an arbitrary non-sin

Let $\{T^\tau_\omega\}$ be the geodesic
$F = F(x,v)$ on $M \times S^1$ let
of $\{T^\tau_\omega\}$.

Theorem 3.2. *Assume th
a Lipschitz function on M
assume, moreover, that* $|F($

$$\frac{1}{S} \int_{M \times S^1} \Bigl| S^t_\omega F(x,v) \ldots$$

for

and for each ε, $0 < \varepsilon \leqslant 0.9$

We do not use the next
immediate consequence of

Theorem 3.3 [2]. (a) *Le
billiard flow in Q to the in
all directions $v \in S^1$.*

(b) *Let ω be a flat struct
flow $\{T^\tau_\omega\}$ to the invariant
$v \in S^1$.*

- a punctured neighbourhood of each singular point is an m-sheeted covering of a punctured neighbourhood of some point in $\mathbb{R}^2$ with covering map that is a translation with respect to each coordinate system in ω; the number m is called the *multiplicity* of a singular point.

Equivalently, a flat structure is a metric of zero curvature on M with finitely many singular points such that we have a conic singularity of angle $2\pi m$ at each singular point, where m is an integer (the multiplicity of the singular point). A flat structure ω gives rise to a geodesic flow on the surface. This flow is not defined everywhere (if a trajectory arrives at a singular point, then it cannot be continued any further), but only on a subset of full-measure (we mean, with respect to the measure μ_ω associated with the metric). The velocity is an integral of the flow, that is, the phase space $M \times S^1$ of the geodesic flow (of unit velocity) is foliated by the invariant surfaces $M \times \{v\}$. We can consider the restriction of the flow to an invariant surface as a flow on M (the flow in the direction v).

Definition 3.2. A *saddle connection* in a flat structure ω is a geodesic segment joining two singular points (maybe coincident) and with no singularities among its interior points.

Let ω be an arbitrary flat structure on M. Let S be the area of M with respect to the measure μ_ω, let s be the length of the shortest saddle connection ω, and let m be the sum of the multiplicities of the singular points in this structure. In what follows we assume that $m > 0$. There is no loss of generality because we can declare an arbitrary non-singular point to be a singular point of multiplicity 1.

Let $\{T_\omega^\tau\}$ be the geodesic flow on M defined by ω. For each measurable function $F = F(x, v)$ on $M \times S^1$ let $S_\omega^t F$ be its mean value over the time t under the action of $\{T_\omega^\tau\}$.

Theorem 3.2. *Assume that $L, E > 0$ and that the function $f_v(x) = F(x, v)$ is a Lipschitz function on M with Lipschitz constant L for each direction $v \in S^1$; assume, moreover, that $|F(x, v)| \leqslant E$ for all $x \in M$ and $v \in S^1$. Then*

$$\frac{1}{S} \int_{M \times S^1} \left| S_\omega^t F(x, v) - \frac{1}{S} \int_M F(y, v)\, d\mu_\omega(y) \right| d\mu_\omega(x)\, d\lambda(v) \leqslant 8\pi E \cdot \varepsilon$$

$$\text{for} \quad t \geqslant \frac{LS}{E} \left(\frac{1}{\varepsilon}\right)^{\left(H(m) \cdot S/s^2 \cdot 1/\varepsilon\right)^{m+5}}$$

and for each ε, $0 < \varepsilon \leqslant 0.999$, where $H(m)$ is the same function as in Theorem 3.1.

We do not use the next result in our proof of the central theorem, but it is an immediate consequence of our discussions in this section.

Theorem 3.3 [2]. (a) *Let Q be a rational polygon. Then the restriction of the billiard flow in Q to the invariant surface $Q \times G(Q)v$ is strictly ergodic for almost all directions $v \in S^1$.*

(b) *Let ω be a flat structure on a surface M. Then the restriction of the geodesic flow $\{T_\omega^\tau\}$ to the invariant surface $M \times \{v\}$ is strictly ergodic for almost all directions $v \in S^1$.*

Remark. Usually, we regard the strict ergodicity as a property of a homeomorphism. In the present case, the strict ergodicity means that there exists a unique normalized Borel measure μ such that the corresponding flow is defined almost everywhere with respect to μ and preserves this measure.

First, we use a construction of Zemlyakov and Katok [1] and reduce the assertion about rational polygons to similar assertions concerning flat structures.

Lemma 3.4. *Theorem* 3.1 *is a consequence of Theorem* 3.2. *Assertion* (a) *of Theorem* 3.3 *is a consequence of assertion* (b).

Proof. Let Q be a rational polygon. We consider the direct product $Q \times G(Q)$ and identify elements of the form (x, g) and (x, gg_a), where x is a point on the side a of Q, g_a is the linear part of the reflection with respect to this side, and $g \in G(Q)$. This done, we obtain a compact connected oriented surface M. The family of charts $\{(U_g, f_g)\}_{g \in G(Q)}$, where $U_g = \operatorname{int} Q \times \{g\}$ and $f_g(x, g) = gx$, can be uniquely complemented to a flat structure ω on M. The singular points of this structure correspond to the vertices of Q. Let m be the sum of the multiplicities of the singular points. Then the sum of the angles at all the singular points is $2\pi m$; on the other hand this sum is equal to $r(Q) \cdot \alpha(Q)$ by construction, therefore $m = m(Q)$. The area S of M with respect to the flat structure ω is $r(Q) \cdot \operatorname{Area}(Q) = S(Q)$. The natural projection of M onto Q maps saddle connections onto generalized diagonals of the same length, and each generalized diagonal is the image of a saddle connection. Hence the length s of the shortest saddle connection ω is $s(Q)$.

There exists a natural projection φ of $M \times S^1$ onto $Q \times S^1$ taking $((x, g), v)$ to $(x, g^{-1}v)$. The map φ transforms the geodesic flow on M into the billiard flow in Q. For all $v \in S^1$, except for finitely many directions invariant with respect to the reflections in $G(Q)$, the map of the surface $M \times \{v\}$ onto $\varphi(M \times \{v\}) = Q \times G(Q)v$ is a homeomorphism, therefore the restrictions of the geodesic flow to $M \times \{v\}$ and of the billiard flow to $Q \times G(Q)v$ are both either strictly ergodic or not. This reduces assertion (a) of Theorem 3.3 to (b).

Let $F(x, v)$ be a function on $Q \times S^1$ satisfying the assumptions of Theorem 3.1. Then the function $\widetilde{F} = F \circ \varphi$ on $M \times S^1$ satisfies the assumptions of Theorem 3.2 (it is essential here that $F(x, v) = 0$ for $x \in \partial Q$). Further,

$$\frac{1}{S(Q)} \sum_{g \in G(Q)} \int_Q F(x, gv)\, d\mu(x) = \frac{1}{S} \int_M \widetilde{F}(x, v)\, d\mu_\omega(x)$$

for each $v \in S^1$. In addition, $S^t_\omega \widetilde{F} = (S^t F) \circ \varphi$. Finally, $\mu \times \lambda\, (\varphi(A)) = \mu_\omega \times \lambda\, (A)$ for each measurable subset A of $M \times S^1$ that is mapped bijectively onto $\varphi(A)$. As a result,

$$\int_{Q \times S^1} \left| S^t F(x, v) - \frac{1}{S(Q)} \sum_{g \in G(Q)} \int_Q F(y, gv)\, d\mu(y) \right| d\mu(x)\, d\lambda(v)$$
$$= \int_{M \times J} \left| S^t_\omega \widetilde{F}(x, v) - \frac{1}{S} \int_M \widetilde{F}(y, v)\, d\mu_\omega(y) \right| d\mu_\omega(x)\, d\lambda(v),$$

where J is an arc of the circle that is fundamental for the action of $G(Q)$ on S^1. The circle is a disjoint union of $r(Q)$ such arcs, therefore

$$\int_{Q\times S^1}\left|S^tF(x,v)-\frac{1}{S(Q)}\sum_{g\in G(Q)}\int_Q F(y,gv)\,d\mu(y)\right|d\mu(x)\,d\lambda(v)$$
$$=\frac{1}{r(Q)}\int_{M\times S^1}\left|S^t_\omega\widetilde{F}(x,v)-\frac{1}{S}\int_M\widetilde{F}(y,v)\,d\mu_\omega(y)\right|d\mu_\omega(x)\,d\lambda(v).$$

Hence Theorem 3.1 is an immediate consequence of Theorem 3.2.

In what follows we consider the geodesic flow corresponding to a flat structure ω on a surface M. Our aim is to prove Theorem 3.12, from which we shall derive Theorem 3.2 and Theorem 3.3(b).

For each $v\in S^1$ we denote by $\{T^\tau_v\}$ the geodesic flow on M in the direction v. Let f be a continuous function on M. For arbitrary $t>0$ and $x\in M$ we set

$$S^t_vf(x)=\frac{1}{t}\int_0^t f(T^\tau_vx)\,d\tau.$$

The average $S^t_vf(x)$ is well defined if T^τ_vx is defined for $0\leqslant\tau\leqslant t$, that is, if the trajectory starting at x in the direction v does not hit a singular point in time t (in particular, x must itself be non-singular). We now introduce the following quantity, which measures the uniformity of the averaging of f under the action of $\{T^\tau_v\}$:

$$M_t=M_t(f,v)=\sup_{\tau\geqslant t}\sup_{x\in M}\left|S^\tau_vf(x)-\frac{1}{S}\int_M f(y)\,d\mu_\omega(y)\right|.$$

Then M_t is a non-increasing continuous function of t for $t>0$.

Proposition 3.5. *Assume that for each $x\in M$ there exists a line segment I containing x, orthogonal to v and such that the trajectories starting at its interior points in the direction v*

(1) *hit singular points no sooner than in time t_0;*
(2) *return to I no sooner than in time $2t_0$;*
(3) *hit singular points or return to I no later than in time Ct_0.*

If the flow on M in the direction v is minimal, then

$$M_{t_0}(f,v)<\frac{LS}{t_0}\quad\text{or}\quad M_{3Ct_0}(f,v)\leqslant M_{t_0}(f,v)\cdot\left(1-\frac{1}{8C}\right)$$

for each Lipschitz function f with Lipschitz constant L.

Proof. We assume without loss of generality that $\int_M f(y)\,d\mu_\omega(y)=0$. Since

$$\int_M S^{t_0}_vf(y)\,d\mu_\omega(y)=\int_M f(y)\,d\mu_\omega(y)=0,$$

there exists $x \in M$ such that $S_v^{t_0} f(x) \leqslant 0$. We now choose the segment I described in the hypotheses of the proposition and containing this point. Condition (1) means that the average $S_v^{t_0} f$ is well defined on the entire segment I (with the possible exception of its end-points). In addition, $S_v^{t_0} f$ is a Lipschitz function on I with constant L; in particular, $S_v^{t_0} f(y) \leqslant L \cdot |I|$ for each $y \in I$. It follows from condition (2) that $2t_0 \cdot |I| \leqslant S$, therefore $S_v^{t_0} f(y) \leqslant \dfrac{LS}{2t_0}$. Assume that $M_{t_0} \geqslant \dfrac{LS}{t_0}$. Then $S_v^{t_0} f(y) \leqslant M_{t_0}/2$.

We now consider an arbitrary trajectory J of length $t > 3Ct_0$ in the direction v. The points of intersection with I partition J into segments $J_1, J_2, \dots, J_n$. Since $\{T_v^\tau\}$ is a minimal flow, it follows from (3) that the lengths of the J_i are at most Ct_0. In particular, $t \leqslant Cnt_0$, therefore $n \geqslant 4$. We now partition each of the J_i, except for the first two and the last segment, into two pieces; the initial segment J_i^1 of length t_0 and the remainder J_i^2. The segments $J_1 + J_2$, J_i^2, $3 \leqslant i \leqslant n-2$, and $J_{n-1}^2 + J_n$ of J are of length at least t_0 by (2), therefore the mean value of f on each of these segments is at most M_{t_0}. On the other hand, the mean value of f on J_i^1 is equal to $S_v^{t_0} f(y)$ for some $y \in I$, therefore it is not larger than $M_{t_0}/2$. Hence the average value of f on J is at most

$$\begin{aligned}
&\frac{1}{t}\left(\frac{M_{t_0}}{2}\, t_0(n-3) + M_{t_0}\big(t - t_0(n-3)\big)\right) \\
&\quad = M_{t_0} \cdot \left(1 - \frac{t_0(n-3)}{2t}\right) \leqslant M_{t_0} \cdot \left(1 - \frac{t_0(n-3)}{2 \cdot Cnt_0}\right) \\
&\quad = M_{t_0} \cdot \left(1 - \left(1 - \frac{3}{n}\right)\frac{1}{2C}\right) \leqslant M_{t_0} \cdot \left(1 - \frac{1}{8C}\right).
\end{aligned}$$

Repeating all these arguments for the function $-f$ we obtain that the mean value of f on J has a lower estimate $-M_{t_0}\left(1 - \dfrac{1}{8C}\right)$. Since the trajectory J was chosen arbitrarily, it follows that

$$M_t \leqslant M_{t_0} \cdot \left(1 - \frac{1}{8C}\right) \quad \text{for } t > 3Ct_0,$$

and therefore we also have $M_{3Ct_0} \leqslant M_{t_0}\left(1 - \dfrac{1}{8C}\right)$.

We find conditions for the applicability of Proposition 3.5 below in Proposition 3.8. Before stating that result we present several definitions and auxiliary statements.

We associate with each geodesic segment I of a flat structure ω (for instance, a saddle connection) the vector depicting I in $\mathbb{R}^2$ and defined up to the change of the direction to the opposite one. For an arbitrary direction $v \in S^1$ let $v(I)$ and $v_\perp(I)$ be the lengths of the projections of this vector onto the direction v and the orthogonal direction, respectively. We denote the length of I by $|I|$.

Definition 3.3. Assume that $t, \varepsilon > 0$. Let $B(t,\varepsilon)$ be the set of directions $v \in S^1$ such that $v(\gamma) \leqslant t$ and $v_\perp(\gamma) \leqslant \varepsilon/t$ for at least one saddle connection γ. We denote the set of directions complementary to $B(t,\varepsilon)$ by $A(t,\varepsilon)$.

In the discussions that follow we use repeatedly the following obvious result.

Lemma 3.6. *Let I be a segment orthogonal to v and containing no singular points in its interior. Consider the trajectories in the direction v starting at all the points of this segment. Assume that at least two of them hit singular points and let t_1 and t_2 be the times of the first and the second hits* ($0 \leqslant t_1 \leqslant t_2$). *Then there exists a saddle connection γ such that $v(\gamma) = t_2 - t_1$, $v_\perp(\gamma) \leqslant |I|$.*

We shall require the following result also in § 4.

Lemma 3.7. *If $\varepsilon \geqslant S$, then $A(t, \varepsilon)$ is empty for each $t > 0$. In particular, $s^2/2 \leqslant S$.*

Proof. For an arbitrary direction v we must find a saddle connection γ such that $v(\gamma) \leqslant t$ and $v_\perp(\gamma) \leqslant \varepsilon/t$. Let x be a singular point of the flat structure ω. We draw a line segment I of length ε/t starting at x in the direction orthogonal to v. If we arrive at a singular point when drawing, then this is the required saddle connection. Otherwise we consider trajectories in the direction v starting at the points of I. In general, there can be several trajectories starting at x; we consider the one that bounds the bundle (the band) of trajectories starting from the other points of I. Let T be the time of the first return to I of a trajectory in this bundle. By Poincaré's recurrence theorem $T \leqslant S/|I| = S/\varepsilon \cdot t \leqslant t$. Two cases are possible now. The first is the case when at least one of the trajectories under consideration hits a singular point at the time T or earlier. Let y be the end-point of this trajectory belonging to I. If $y = x$, then we obtain a saddle connection, which is the required one. For $y \neq x$ a required saddle connection exists by Lemma 3.6. The second is the case when the trajectories in the bundle do not arrive at singular points in time T. Then it is easy to see that the trajectory from x intersects I at the time T at some point z, $z \neq x$. Hence the trajectory from z in the direction opposite to v hits a singular point at time $T \leqslant t$, so that we can again find the required saddle connection using Lemma 3.6.

Thus, $A(t, \varepsilon)$ is empty for $\varepsilon \geqslant S$. In particular, the set $A(\sqrt{S}, S)$ is empty, that is, for each direction v there exists a saddle connection γ such that $v(\gamma) \leqslant \sqrt{S}$ and $v_\perp(\gamma) \leqslant S/\sqrt{S} = \sqrt{S}$. Its length is at most $\sqrt{2S}$; on the other hand $|\gamma| \geqslant s$. Hence $s \leqslant \sqrt{2S}$ and $s^2/2 \leqslant S$.

Proposition 3.8. *Let v be a direction such that there exists no saddle connection parallel to v. If v belongs to both sets $A(2t_0, \varepsilon)$ and $A(24t_0 \cdot (3i+1)(S/\varepsilon)^{i+1}, \varepsilon)$, $0 \leqslant i \leqslant m+2$, for some $\varepsilon > 0$, then all the assumptions of Proposition 3.5 hold with constant*

$$C = 24(3m+7)(S/\varepsilon)^{m+4}.$$

Proof. We prove this proposition in three steps. First, for an arbitrary point $x \in M$ we construct a line segment I orthogonal to v and of length $\Delta = \varepsilon/(8t_0)$ with end-point x. This segment I has the following property: the trajectories in the direction v from its interior points do not hit singular points and do not return to I before the time $2t_0$. The second step is to find a subsegment I_0 of I of length at least $\Delta/3$ such that the trajectories from the end-points of I_0 hit singular points no later than at the time $T = 24t_0 \cdot 3(S/\varepsilon)^2$. The third step is the proof of the fact that the trajectories from the points of I_0 in the direction v hit singular points or return to I_0 by the time Ct_0, where C is as in the statement.

Thus, let x be an arbitrary point of M. We draw line segments with end-point x and of length $2\Delta = \varepsilon/(4t_0)$ in the two directions orthogonal to v. If one of the segments arrives at a singular point, then we do not continue it any more. If x is itself a singular point, then we draw only one segment (any one). We denote the union of these segments by I'; it cannot be a saddle connection or a closed trajectory, for otherwise there exists a saddle connection orthogonal to v of length at most $|I'| \leqslant 4\Delta = \varepsilon/(2t_0)$, which contradicts the condition $v \in A(2t_0, \varepsilon)$. Hence I' is a segment with no singularities as interior points and at least one of its end-points is non-singular. Since $|I'| \leqslant \dfrac{\varepsilon}{2t_0}$, it follows by Lemma 3.6 and the condition $v \in A(2t_0, \varepsilon)$ that, of all the trajectories starting at the points of I' in the direction v, at most one hits a singular point by time $2t_0$. Hence one of the two subsegments of I' of length 2Δ separated by x has the following property: the trajectories from its interior points in the direction v do not arrive at singular points by time $2t_0$. Let I'' be this segment, let I be its half with end-point x, let I''' be the other half, and let y be the middle point of I''. We claim that I is the segment required at the first step of the proof, that is, the trajectories from its interior points in the direction v do not return to I by the time $2t_0$. For assume that this condition is violated and the first return to I occurs at some time $t \leqslant 2t_0$. Then the trajectories from the interior points of I'' continue to make up a single bundle at time t, therefore either all the trajectories starting at the points of I or all the trajectories starting at I''' intersect I'' at this time. Hence the trajectories from y in the direction v and in the opposite direction intersect I'' at the time t at some points z_+ and z_-, respectively. These points, z_+ and z_-, are distinct since otherwise we obtain a closed trajectory parallel to v, in which case there must also exist a saddle connection parallel to v. The point y is at the middle of the segment z_-z_+. Further, there are no saddle connections parallel to v, and therefore the flow on M in the direction v is minimal (see, for instance, [1]). In particular, there is at least one trajectory starting at z_-z_+ in the direction v and hitting a singular point. Let t' be the time of the first hit, and let z_1 be the end-point of the corresponding trajectory in z_-z_+. Then z_1 lies on the segment yz_+ because, by construction, all the trajectories from the points of z_-y intersect the segment yz_+ at the time t and do not hit singular points before this time. Let $z_2 \in z_-y$ be the end-point of the trajectory that arrives at z_1 at the time t (the distance between z_1 and z_2 is equal to the length of yz_+). At time $t'+t$ the same trajectory arrives at a singular point. By Lemma 3.6 there exists a saddle connection γ such that $v(\gamma) \leqslant t \leqslant 2t_0$, $v_\perp(\gamma) \leqslant |z_-z_+| \leqslant \varepsilon/(4t_0)$. However, this contradicts the condition $v \in A(2t_0, \varepsilon)$.

We now proceed to the second step in our proof. Let I_1 be an arbitrary subsegment of I of length $\Delta/3$. We claim that trajectories starting at I_1 in the direction v hit some singular point not later than at time $T = 24t_0 \cdot 3(S/\varepsilon)^2$. (We note that the flow is minimal in the direction v and therefore at least one singular point will be hit.) The first return to I_1 of an interior point of this segment under the action of the flow in the direction v occurs at some instant t_1' that is not later than $t_1 = \dfrac{S}{\Delta/3} = 24t_0 \cdot S/\varepsilon$. If we have hit a singular point before that, then there is nothing to prove, for $\varepsilon < S$ by Lemma 3.7 and therefore $t_1 \leqslant T$. Otherwise, all the points of I_1 are still moving as a single bundle at the time t_1'. However, not all of them return to I_1 at this moment (for otherwise no singular point whatsoever can be hit). Hence the situation is as follows: the points returning to I_1 belong to some subsegment with end-points w_0 and w_1, where w_0 is an interior point and w_1 an end-point of I_1. Upon their return, all the points of w_0w_1 are shifted by some distance from their initial position so that the point w_0 is taken to the point w_2, the end-point of I_1 distinct from w_1. We claim that the length Δ_1' of w_2w_0 is at least $\Delta_1 = \dfrac{\varepsilon}{2S} \cdot \dfrac{\Delta}{3}$. If $\Delta_1' \geqslant |I_1|/2$, then there is nothing to prove, since $|I_1|/2 = 1/2 \cdot \Delta/3 \geqslant \Delta_1$. Otherwise we consider the points $u_1, u_2 \in I_1$ that are at the distances Δ_1' and $2\Delta_1'$, respectively, from w_1. In time t_1', the flow in the direction v moves all the points in the segment u_1w_1 into u_2u_1. Hence the time between the first and the second hits of singular points by trajectories starting at the segment u_2w_1 in this direction is at most t_1'. Since $v \in A(t_1, \varepsilon)$, it follows by Lemma 3.6 that $|u_2w_1| > \varepsilon/t_1 = \varepsilon/S \cdot \Delta/3$, that is, $\Delta_1' = |u_2w_1|/2 > \Delta_1$.

The first return to I_1 of interior points in w_2w_0 moving under the action of the flow in the direction v occurs at some time t_2', which is not later than the time $t_2 = S/\Delta_1 = 24t_0 \cdot 2(S/\varepsilon)^2$ and not before t_1'. All the returned points will be (at the moment of the return) at distances smaller than Δ_1' from the end-point w_1 of I_1 since otherwise they would pass the segment w_0w_1 before the time t_1'. Hence if no singular point has been hit yet, then some point w in w_2w_0 arrives at w_1 at this moment. All the points of w_2w return to I_1 together with it. At the time $t_1' + t_2'$ the point w returns to I_1 again, and if no singular point has been hit yet, then all the points of ww_0 return to I_1 at this moment. Thus, by the time $t_1' + t_2'$, either all points in I_1 manage to return to this segment or at least one of them has hit a singular point. Since a singular point must be hit sooner or later, the second alternative holds, that is, the first singular point is hit not later than the time $t_1' + t_2' \leqslant t_1 + t_2$, which is not later than time T.

We now partition I into three equal parts. In each of the extreme parts we choose points such that, moving under the action of the flow in the direction v, they hit singular points not later than the time T. Let I_0 be the segment with end-points at these points. Its length is at least $\Delta/3$, so that this is the required segment.

We now proceed to the third step in the proof. Let $p_1, \dots, p_k$ be the interior points of I_0 that hit singular points under the action of the flow in the direction v before returning to I_0 or hit the end-points of I_0 when they return first. The number of points hitting singular points before returning to I_0 is not larger than the sum of the multiplicities of the singular points, therefore $k \leqslant m + 2$. Let $\tau_1, \dots, \tau_k$ be the times of the arrival of $p_1, \dots, p_k$ at singular points or the end-points of I_0. We assume that the points $p_1, \dots, p_k$ are ordered so that $\tau_1 \leqslant \tau_2 \leqslant \dots \leqslant \tau_k$. Let δ_i $(0 \leqslant i \leqslant k)$ be the length of the smallest segment in the partitioning of I_0 by $p_1, \dots, p_i$. Using induction on i we shall now prove that $\tau_i \leqslant 24t_0 \cdot (3i-2)(S/\varepsilon)^{i+1}$ for $1 \leqslant i \leqslant k$ and $\delta_i \geqslant \dfrac{\varepsilon}{24t_0 \cdot (3i+1)(S/\varepsilon)^{i+1}}$ for $0 \leqslant i \leqslant k$. For $i = 0$ we need only prove that $\delta_0 = |I_0| \geqslant \dfrac{\varepsilon}{24t_0 \cdot S/\varepsilon}$. But indeed, $|I_0| \geqslant \Delta/3 = \dfrac{\varepsilon}{24t_0}$, which is not less than $\dfrac{\varepsilon}{24t_0 \cdot S/\varepsilon}$. Assume now that $0 < i \leqslant k$ and that we have already proved the inequality $\delta_{i-1} \geqslant \dfrac{\varepsilon}{24t_0 \cdot (3i-2)(S/\varepsilon)^i}$. Let J_i be the segment in the partitioning of I_0 by the points $p_1, \ldots, p_{i-1}$ that contains p_i. Since $|J_i| \geqslant \delta_{i-1}$, the time T_i of the first return to I_0 of trajectories starting from interior points of J_i in the direction v is $S/\delta_{i-1} \leqslant 24t_0 \cdot (3i-2)(S/\varepsilon)^{i+1}$ at the latest. Some of these trajectories do not return to interior points of I_0 at time T_i (this holds, for instance, for the trajectory starting at p_i). However, this is possible only in the case when some trajectory has already hit a singular point or an end-point of I_0 by this time, that is, when $\tau_i \leqslant T_i$, so that we obtain the required estimate of τ_i. Next, let J_i' be the shortest segment in the partitioning of I_0 by the points $p_1, \ldots, p_i$. Then the trajectories starting at the end-points of J_i' hit singular points at the time

$$\tau_i + T \leqslant 24t_0 \cdot (3i-2)(S/\varepsilon)^{i+1} + 24t_0 \cdot 3(S/\varepsilon)^2 \leqslant 24t_0 \cdot (3i+1)(S/\varepsilon)^{i+1}$$

at the latest. By the condition $v \in A(24t_0 \cdot (3i+1)(S/\varepsilon)^{i+1}, \varepsilon)$ and Lemma 3.6 we obtain

$$\delta_i = |J_i'| > \frac{\varepsilon}{24t_0 \cdot (3i+1)(S/\varepsilon)^{i+1}},$$

as required.

Let J be one of the segments in the partitioning of I_0 by $p_1, \ldots, p_k$. Under the action of the flow in the direction v the interior points of J return to I_0 without having hit its end-points or singular points. Hence they return all at the same time. Since

$$|J| \geqslant \delta_k \geqslant \frac{\varepsilon}{24t_0 \cdot (3k+1)(S/\varepsilon)^{k+1}},$$

the time of return is, at the latest,

$$S/|J| \leqslant S/\delta_k \leqslant 24t_0 \cdot (3k+1)(S/\varepsilon)^{k+2} \leqslant 24t_0 \cdot (3m+7)(S/\varepsilon)^{m+4}.$$

Hence the trajectories starting at the points of I_0 in the direction v arrive at singular points or come back to I_0 by the time $24t_0 \cdot (3m+7)(S/\varepsilon)^{m+4}$. Since the flow in the direction v is minimal, the same assertion holds for the segment I containing I_0.

Hence I satisfies assumptions (1)–(3) of Proposition 3.5 with the required constant C.

For an arbitrary $\varepsilon > 0$ we denote the quantity $24(3m+7)(S/\varepsilon)^{m+4}$ by $C(\varepsilon)$. Let $B_1(t, \varepsilon)$ be the union of the $m+4$ sets $B(2t, \varepsilon)$ and $B(24t(3i+1)(S/\varepsilon)^{i+1}, \varepsilon)$, $0 \leqslant i \leqslant m+2$. For each integer $n > 0$ let $B_2(t, \varepsilon, n)$ be the set of directions belonging to at least n of the $2n$ sets $B_1(t, \varepsilon), B_1(3C(\varepsilon)t, \varepsilon), \ldots, B_1((3C(\varepsilon))^{2n-1}t, \varepsilon)$.

Lemma 3.9. *Let f be a Lipschitz function with Lipschitz constant L on the surface M and assume that $|f(x)| \leqslant E$ for all $x \in M$. If $t \geqslant LS/E$, then for each direction v that is not parallel to a saddle connection and does not belong to $B_2(t, \varepsilon, n)$ we have*

$$M_{(3C(\varepsilon))^{2n}t}(f, v) \leqslant 2E \cdot \left(1 - \frac{1}{8C(\varepsilon)}\right)^n.$$

Proof. By assumption, v does not belong to the sets

$$B_1\big((3C)^{i_0}t, \varepsilon\big),\ B_1\big((3C)^{i_1}t, \varepsilon\big),\ \ldots,\ B_1\big((3C)^{i_{n-1}}t, \varepsilon\big),$$

where $i_0, i_1, \ldots, i_{n-1}$ are certain indices, $0 \leqslant i_0 < i_1 < \cdots < i_{n-1} \leqslant 2n-1$. We claim that

$$M_{(3C)^{i_j}t} \leqslant 2E \cdot \left(1 - \frac{1}{8C}\right)^j \quad \text{for } 0 \leqslant j \leqslant n,$$

where $i_n = 2n$ by definition. We prove this by induction on j. For $j = 0$ this inequality is obvious since $M_{t'}(f, v') \leqslant 2E$ for each $t' > 0$ and each direction v'. Next, assume that the inequality holds for some value of j, $0 \leqslant j < n$. Since the direction of v is not parallel to any saddle connection and $v \notin B_1((3C)^{i_j}t, \varepsilon)$, it follows by Propositions 3.5 and 3.8 that

$$\begin{aligned} M_{(3C)^{i_j+1}t} &\leqslant \max\left(\frac{LS}{(3C)^{i_j}t}, M_{(3C)^{i_j}t}\left(1 - \frac{1}{8C}\right)\right) \\ &\leqslant \max\left(\frac{LS}{(3C)^{j}t}, 2E\left(1 - \frac{1}{8C}\right)^{j+1}\right). \end{aligned}$$

By Lemma 3.7 we have $\varepsilon < S$, therefore $C = C(\varepsilon) > 24(3m+7) \geqslant 240$ and, in particular, $\frac{1}{3C} < 1 - \frac{1}{8C}$, $2\left(1 - \frac{1}{8C}\right) > 1$. Hence

$$\frac{LS}{(3C)^j t} \leqslant \frac{E}{(3C)^j} \leqslant E\left(1 - \frac{1}{8C}\right)^j \leqslant 2E\left(1 - \frac{1}{8C}\right)^{j+1},$$

so that

$$M_{(3C)^{i_j+1}t} \leqslant 2E\left(1 - \frac{1}{8C}\right)^{j+1}.$$

Since $i_j + 1 \leqslant i_{j+1}$, it follows that $M_{(3C)^{i_{j+1}}t} \leqslant M_{(3C)^{i_j+1}t}$, which completes the inductive step. The assertion of the lemma follows from the above inequality with $j = n$.

In what follows we find estimates for the measures of the sets $B(t, \varepsilon)$, $B_1(t, \varepsilon)$, and $B_2(t, \varepsilon, n)$. Here we shall use Theorem 4.1 proved in § 4. The arguments used in the proof of Lemma 3.10 are due to Boshernitzan [5].

Lemma 3.10. *We have the inequality* $\lambda(B(t, \varepsilon)) \leqslant C_1 \cdot \varepsilon$, *where* $C_1 = 2\pi\, h(m)/s^2$ *with* $h(m) = (400m)^{(2m)^{2m}}$ *for* $m > 1$ *and* $h(1) = (3 \cdot 2^7)^6$ (λ *is Lebesgue measure on* S^1).

Proof. The set of directions $B(\varepsilon/t, \varepsilon)$ is the result of a rotation of $B(t, \varepsilon)$ through the angle $\pi/2$, therefore the measures of these sets are the same. Hence we can assume without loss of generality that $t \geqslant \varepsilon/t$.

Let γ be a saddle connection and let v be a direction such that $v(\gamma) \leqslant t$ and $v_\perp(\gamma) \leqslant \varepsilon/t$. We can estimate the length of γ and the angle between γ and v (assuming that $0 \leqslant \angle(\gamma, v) \leqslant \pi/2$) as follows:

$$|\gamma| \leqslant v(\gamma) + v_\perp(\gamma) \leqslant t + \varepsilon/t \leqslant 2t,$$

$$\angle(\gamma, v) \leqslant \frac{\pi}{2}\sin\angle(\gamma, v) = \frac{\pi}{2} \cdot \frac{v_\perp(\gamma)}{|\gamma|} \leqslant \frac{\pi}{2} \cdot \frac{\varepsilon}{t|\gamma|}.$$

The measure of the set of directions making an angle at most $\frac{\pi}{2}\cdot\frac{\varepsilon}{t|\gamma|}$ with γ is $\min\left(2\pi, 4\cdot\frac{\pi}{2}\frac{\varepsilon}{t|\gamma|}\right)\leqslant\frac{2\pi\varepsilon}{t|\gamma|}$. In view of the above estimates,

$$\lambda(B(t,\varepsilon))\leqslant\frac{2\pi\varepsilon}{t}\sum_{\gamma}|\gamma|^{-1},$$

where the sum is taken over all the saddle connections of length at most $2t$.

Let $\gamma_1,\gamma_2,\dots$ be the saddle connections in a flat structure ω indexed so that $|\gamma_1|\leqslant|\gamma_2|\leqslant\cdots$. By Theorem 4.1 there exist at most $h(m)\,(l/s)^2$ saddle connections of length not larger than l. Hence $n\leqslant h(m)\cdot(|\gamma_n|/s)^2$ and $|\gamma_n|\geqslant\frac{s}{\sqrt{h(m)}}\sqrt{n}$ for each index n. Further, if $|\gamma_n|\leqslant 2t$, then $n\leqslant h(m)\,(2t/s)^2$, so that setting $N(t)=h(m)\,(2t/s)^2$ we obtain

$$\sum_{|\gamma|\leqslant 2t}|\gamma|^{-1}\leqslant\sum_{n\leqslant N(t)}|\gamma_n|^{-1}\leqslant\frac{\sqrt{h(m)}}{s}\sum_{n\leqslant N(t)}\frac{1}{\sqrt{n}}$$
$$\leqslant\frac{\sqrt{h(m)}}{s}\int_0^{N(t)}\frac{dx}{\sqrt{x}}=\frac{\sqrt{h(m)}}{s}\,\frac{\sqrt{N(t)}}{2}=\frac{h(m)}{s^2}\,t.$$

As a result, $\lambda(B(t,\varepsilon))\leqslant 2\pi\varepsilon/t\cdot h(m)/s^2\cdot t=2\pi\varepsilon/s^2\cdot h(m)$, as required.

Corollary 3.11. *We have* $\lambda(B_1(t,\varepsilon))\leqslant(m+4)C_1\varepsilon$, $\lambda(B_2(t,\varepsilon,n))\leqslant 2(m+4)C_1\varepsilon$, *where* C_1 *is as in Lemma* 3.10.

Proof. The first estimate is obvious since $B_1(t,\varepsilon)$ is the union of $m+4$ sets of the form $B(t',\varepsilon)$. Next, for each direction v we denote by $g(v)$ the number of sets among $B_1(t,\varepsilon),B_1(3C(\varepsilon)t,\varepsilon),\dots,B_1((3C(\varepsilon))^{2n-1}t,\varepsilon)$ that contain v. The function $g(v)$ is the sum of the characteristic functions of $2n$ sets of the form $B_1(t',\varepsilon)$, therefore

$$\int_{S^1}g(v)\,d\lambda(v)\leqslant 2n(m+4)C_1\varepsilon.$$

Since $g(v)\geqslant n$ for $v\in B_2(t,\varepsilon,n)$, we obtain $n\cdot\lambda(B_2(t,\varepsilon,n))\leqslant 2n(m+4)C_1\varepsilon$, therefore $\lambda(B_2(t,\varepsilon,n))\leqslant 2(m+4)C_1\varepsilon$.

The following theorem sums up the results obtained earlier in this section.

Theorem 3.12. *Let* ε_1 $(0<\varepsilon_1\leqslant 1)$, ε_2 $(0<\varepsilon_2\leqslant 0.999)$, L, *and* E *be positive constants. Then there exists a set of directions* B *dependent on all these constants such that its measure is at most* $2\pi\varepsilon_1$ *and for each direction* $v\notin B$ *and each Lipschitz function* f *on* M *that has Lipschitz constant* L *and is not larger than* E *in absolute value we have*

$$M_t(f,v)\leqslant 2E\cdot\varepsilon_2\quad\text{for } t\geqslant\frac{LS}{E}\left(\frac{1}{\varepsilon_2}\right)^{\left(H(m)\cdot S/s^2\cdot 1/\varepsilon_1\right)^{m+5}},$$

where $H(1)=2^{60}$ *and* $H(m)=(500m)^{(2m)^{2m}}$ *for* $m>1$.

Proof. We set $\varepsilon = \dfrac{s^2}{2(m+4)\,h(m)}\,\varepsilon_1$ (here $h(m)$ is as in Lemma 3.10). Also, let $n = \left\lceil \dfrac{\log 1/\varepsilon_2}{\log\left(1-\frac{1}{8C(\varepsilon)}\right)^{-1}} \right\rceil$ (here $\lceil x \rceil$ is the smallest integer larger or equal to x) and let $T = \dfrac{LS}{E}\,(3C(\varepsilon))^{2n}$. By Corollary 3.11 we have $\lambda(B_2(T,\varepsilon,n)) \leqslant 2\pi\varepsilon_1$. If the direction v does not belong to $B_2(T,\varepsilon,n)$ or to the countable set of directions parallel to the saddle connections, then $M_T(f,v) \leqslant 2E\left(1-\dfrac{1}{8C}\right)^n$ by Lemma 3.9, and the latter does not exceed $2E\cdot\varepsilon_2$ by our choice of n. Thus, the proof of the theorem reduces to the verification of the inequality

$$T \leqslant \frac{LS}{E}\left(\frac{1}{\varepsilon_2}\right)^{\left(H(m)\cdot S/s^2\cdot 1/\varepsilon_1\right)^{m+5}}.$$

We now find a lower estimate for $C = C(\varepsilon)$. We have

$$C = 24(3m+7)\left(2(m+4)\,h(m)\cdot S/s^2\cdot 1/\varepsilon_1\right)^{m+4},$$

where $1/\varepsilon_1 \geqslant 1$, while $S/s^2 \geqslant 1/2$ by Lemma 3.7, therefore

$$C \geqslant 24(3m+7)\left((m+4)\,h(m)\right)^{m+4},$$

which is obviously larger than 1000. Hence $\left(1-\dfrac{1}{8C}\right)^{-1} \leqslant 1.001 < 1/\varepsilon_2$, therefore

$$n < \frac{\log 1/\varepsilon_2}{\log\left(1-\frac{1}{8C}\right)^{-1}} + 1 < 2\,\frac{\log 1/\varepsilon_2}{\log\left(1-\frac{1}{8C}\right)^{-1}}.$$

Next,

$$\log\left(1-\frac{1}{8C}\right)^{-1} = \log\left(1+\frac{1}{8C-1}\right) \geqslant \frac{1}{8C-1} - \frac{1}{2}\left(\frac{1}{8C-1}\right)^2 = \frac{1}{8C}\left(1+\frac{8C-2}{2(8C-1)^2}\right) > \frac{1}{8C}.$$

As a result, $n \leqslant 2\cdot 8C\cdot\log 1/\varepsilon_2$ and

$$T \leqslant \frac{LS}{E}\,(3C)^{32C\cdot\log 1/\varepsilon_2} = \frac{LS}{E}\,(1/\varepsilon_2)^{32C\cdot\log(3C)}.$$

The function $g(x) = x^{1/x}$ decreases for $x \geqslant e$, therefore $(m+4)^{\frac{1}{m+4}} \leqslant 5^{1/5}$ and $(3m+7)^{\frac{1}{m+4}} < (3m+12)^{\frac{1}{3m+12}\cdot 3} \leqslant 15^{\frac{1}{15}\cdot 3} = 15^{1/5}$. Next, $(32\cdot 24)^{\frac{1}{m+4}} \leqslant (32\cdot 24)^{1/5}$. Hence $\left(32\cdot 24(3m+7)(m+4)\right)^{\frac{1}{m+4}} \leqslant (32\cdot 24\cdot 15\cdot 5)^{1/5} < 10$. Consequently,

$$3C = 3\cdot 24(3m+7)\left(2(m+4)\,h(m)\,S/s^2\cdot 1/\varepsilon_1\right)^{m+4} \leqslant \left(20(m+4)\,h(m)\,S/s^2\cdot 1/\varepsilon_1\right)^{m+4},$$

therefore

$$
\begin{aligned}
32C \cdot \log(3C) &\leqslant 32C \cdot (m+4) \cdot \log\bigl(20(m+4)\, h(m)\, S/s^2 \cdot 1/\varepsilon_1\bigr) \\
&= 32 \cdot 24(3m+7)(m+4)\bigl(2(m+4)\, h(m)\, S/s^2 \cdot 1/\varepsilon_1\bigr)^{m+4} \\
&\qquad \times \log\bigl(20(m+4)\, h(m)\, S/s^2 \cdot 1/\varepsilon_1\bigr) \\
&\leqslant \bigl(20(m+4)h(m)S/s^2 \cdot 1/\varepsilon_1\bigr)^{m+4} \log\bigl(20(m+4)h(m)S/s^2 \cdot 1/\varepsilon_1\bigr) \\
&\leqslant \bigl(20(m+4)h(m)S/s^2 \cdot 1/\varepsilon_1\bigr)^{m+5}.
\end{aligned}
$$

For $m = 1$,

$$20(m+4)h(m) = 20 \cdot 5 \cdot (3 \cdot 2^7)^6 < 2^{60} = H(1).$$

On the other hand if $m > 1$, then $20^{(2m)^{-2m}} \leqslant 20^{4^{-4}} < 20^{1/120}$ and

$$(m+4)^{(2m)^{-2m}} = (m+4)^{\frac{1}{m+4} \cdot \frac{m+4}{(2m)^{2m}}} \leqslant (6^{1/6})^{\frac{3m}{(2m)^4}} < (6^{1/6})^{1/32} < 6^{1/120}.$$

Hence $\bigl(20(m+4)\bigr)^{(2m)^{-2m}} < 120^{1/120} < 16^{1/16} = 2^{1/4} < 5/4$ and

$$20(m+4)h(m) = 20(m+4)(400m)^{(2m)^{2m}} < (5/4 \cdot 400m)^{(2m)^{2m}} = H(m).$$

As a result,

$$32C \cdot \log(3C) \leqslant \bigl(20(m+4)h(m)S/s^2 \cdot 1/\varepsilon_1\bigr)^{m+5} < \bigl(H(m) \cdot S/s^2 \cdot 1/\varepsilon_1\bigr)^{m+5}$$

for each value of m, therefore

$$T \leqslant \frac{LS}{E}\left(\frac{1}{\varepsilon_2}\right)^{32C \cdot \log(3C)} < \frac{LS}{E}\left(\frac{1}{\varepsilon_2}\right)^{\left(H(m) \cdot S/s^2 \cdot 1/\varepsilon_1\right)^{m+5}},$$

as required.

Proof of Theorem 3.2. We have the following estimate:

$$\frac{1}{S}\int_{M \times S^1}\left| S_\omega^t F(x,v) - \frac{1}{S}\int_M F(y,v)\, d\mu_\omega(y)\right| d\mu_\omega(x)\, d\lambda(v) \leqslant \int_{S^1} M_t(f_v, v)\, d\lambda(v).$$

We set $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Let B be the set of directions corresponding to the parameters $\varepsilon_1, \varepsilon_2, L$, and E, the existence of which is established in Theorem 3.12. Then we have $M_t(f_v, v) \leqslant 2E\varepsilon$ for

$$t \geqslant \frac{LS}{E}(1/\varepsilon)^{\left(H(m) \cdot S/s^2 \cdot 1/\varepsilon\right)^{m+5}},$$

provided that $v \notin B$. If, on the other hand, $v \in B$, then we can use the simplest estimate $M_t(f_v, v) \leqslant 2E$. Since $\lambda(B) \leqslant 2\pi\varepsilon$, it follows for these values of t that

$$\int_{S^1} M_t(f_v, v)\, d\lambda(v) \leqslant 2E \cdot \lambda(B) + \bigl(2\pi - \lambda(B)\bigr) \cdot 2E\varepsilon \leqslant 8\pi E \cdot \varepsilon.$$

Proof of Theorem 3.3(b). Le
functions on M that is den
to the uniform norm. By T
$M_t(f_n, v) \not\to 0$ as $t \to \infty$ ha
each index n if v is outside
if f and $\widetilde{f}$ are functions suc
$|M_t(f,v) - M_t(\widetilde{f},v)| \leqslant 2\varepsilon$ fo

We now consider an arbi
on M can be uniformly appr
therefore $M_t(f,v) \to 0$ as t -
is the unique (up to a scala
to the flow in the direction

4. Quadratic grov

Let ω be a flat structure
of multiplicities of the singu
saddle connection.

The aim of this section is
connections of length not lar

Theorem 4.1. *We have th*
and $h(m) = (400m)^{(2m)^{2m}}$ *f*

Remark. Masur proved in
flat structure. We have bor
number of saddle connectio
essentially different way fror
we make no use of the lang
of [2] and [4]. Instead, we u
for the derivation of effectiv

Definition 4.1. A *comple*
bounded by pairwise disjoi
to be disjoint if they have
bounding K may include *c*
both sides.

An *ω-triangle* is a triangl
connections as sides that co

We now state the geom
required for what follows.

Proposition 4.2. (a) *Each*
augmented to an ω-triangul

(b) *The number of ω-tria*
the number of saddle conne

Proof of Theorem 3.3(b). Let $f_1, f_2, \dots, f_n, \dots$ be a countable family of Lipschitz functions on M that is dense in the space of continuous functions with respect to the uniform norm. By Theorem 3.12 the set B_n of the directions v such that $M_t(f_n, v) \not\to 0$ as $t \to \infty$ has measure zero. Hence $M_t(f_n, v) \to 0$ as $t \to \infty$ for each index n if v is outside a certain subset $B \subset S^1$ of measure zero. Further, if f and $\widetilde{f}$ are functions such that $|f(x) - \widetilde{f}(x)| \leqslant \varepsilon$ for all $x \in M$, then, clearly, $|M_t(f, v) - M_t(\widetilde{f}, v)| \leqslant 2\varepsilon$ for each $t > 0$ and each $v \in S^1$.

We now consider an arbitrary direction $v \notin B$. Each continuous function f on M can be uniformly approximated by functions in the sequence $f_1, \dots, f_n, \dots$, therefore $M_t(f, v) \to 0$ as $t \to \infty$ by the above. In view of the ergodic theorem, μ_ω is the unique (up to a scalar factor) measure on M that is invariant with respect to the flow in the direction v.

4. Quadratic growth in the number of saddle connections

Let ω be a flat structure on a compact connected surface M. Let m be the sum of multiplicities of the singular points of ω and let s be the length of the shortest saddle connection.

The aim of this section is to obtain an estimate of the number $N(L)$ of the saddle connections of length not larger than L in ω. The definitive result here is as follows.

Theorem 4.1. *We have the estimate* $N(L) \leqslant h(m) \cdot \left(\frac{L}{s}\right)^2$, *where* $h(1) = (3 \cdot 2^7)^6$ *and* $h(m) = (400m)^{(2m)^{2m}}$ *for* $m > 1$.

Remark. Masur proved in [4] that $N(L) = O(L^2)$ as $L \to \infty$ for an arbitrary flat structure. We have borrowed the entire inductive scheme of evaluation of the number of saddle connections from [4]. However, we implement this scheme in an essentially different way from the original one. In addition, we must point out that we make no use of the language of the Teichmüller theory, which is characteristic of [2] and [4]. Instead, we use the language of projections, which is more suitable for the derivation of effective estimates.

Definition 4.1. A *complex* K is either a saddle connection or a subdomain of M bounded by pairwise disjoint saddle connections (two saddle connections are said to be disjoint if they have no common interior points). The saddle connections bounding K may include *cuts*, that is, saddle connections having the complex on both sides.

An ω-*triangle* is a triangle on M with vertices at singular points and with saddle connections as sides that contains no singular points in its interior.

We now state the geometric properties of complexes and saddle connections required for what follows.

Proposition 4.2. (a) *Each collection of pairwise disjoint saddle connections can be augmented to an* ω-*triangulation of* M, *that is, to a partition of it into* ω-*triangles.*

(b) *The number of* ω-*triangles in an arbitrary* ω-*triangulation is equal to* $2m$ *and the number of saddle connections is equal to* $3m$.

(c) *An arbitrary complex K can be partitioned into ω-triangles by saddle connections; the number $d(K)$ of the resulting ω-triangles is independent of the partitioning and $0 \leqslant d(K) \leqslant 2m$.*

Proof. Assertion (a) is obvious, while (c) is a consequence of (a) and (b). We now prove (b). Let m_2 and m_3 be the numbers of the saddle connections and ω-triangles, respectively, in some ω-triangulation of M. The total sum Σ of all the angles of the ω-triangles is $m_3\pi$. On the other hand, the sum of all the angles at a singular point of multiplicity n is $2\pi n$, therefore $\Sigma = 2\pi m$ and $m_3 = 2m$. The equality $m_2 = 3m$ is a consequence of the obvious relation $2m_2 = 3m_3$.

We recall that we have denoted the lengths of the projections of a saddle connection γ (more precisely, of the vector depicting it in $\mathbb{R}^2$) in the direction v and the orthogonal direction by $v(\gamma)$ and $v_\perp(\gamma)$, respectively, and we denote the length of γ itself by $|\gamma|$. If K is a complex, then let $v(K)$ (or $v_\perp(K)$, or $|K|$) be the largest of the quantities $v(\gamma)$ (or $v_\perp(\gamma)$, or $|\gamma|$, respectively) corresponding to all the saddle connections γ at the boundary of K.

Definition 4.2. Let l, δ, and C be positive numbers and let d be an integer, $0 \leqslant d \leqslant 2m$. We say that a saddle connection γ is *(l, δ, C, d)-close* to the direction v if

$$v(\gamma) \leqslant (C+2)^d\, l \quad \text{and} \quad v_\perp(\gamma) \leqslant (C+2)^d\, \delta/l.$$

We say that γ is *(l, δ, C, d)-insulated* relative to v if it is (l, δ, C, d)-close to v and each saddle connection $\widetilde{\gamma} \neq \gamma$ satisfying the conditions

$$v(\widetilde{\gamma}) \leqslant C(C+2)^d\, l \quad \text{and} \quad v_\perp(\widetilde{\gamma}) \leqslant C(C+2)^d\, \delta/l,$$

is disjoint from γ.

Let K be a complex containing a saddle connection γ. We modify the definition of an insulated saddle connection and request that the saddle connections $\widetilde{\gamma}$ in this definition also lie in K. Then we obtain a definition of a weaker property, which we call the *insulation of γ within K*. The former definition relates to the case when K is the entire surface M (with no cuts).

Definition 4.3. We say that a complex K is *(l, δ, C)-close* to the direction v if each of its boundary saddle connections is $(l, \delta, C, d(K))$-close to v.

In a similar way we can define the *(l, δ, C)-insulation* of a complex K relative to a direction v, both an ordinary one and within a complex $\widetilde{K}$ containing K.

Lemma 4.3. *Let K be a subcomplex of a larger complex $\widetilde{K}$ and let γ be a saddle connection at the boundary of K that is not a cut. Assume that K is (l, δ, C)-close to some direction v, but at the same time γ is not $(l, \delta, C, d(K))$-insulated relative to v within $\widetilde{K}$. Then there exists a complex K_1 such that it is (l, δ, C)-close to v and*

(a) $K \subset K_1 \subset \widetilde{K}$;

(b) *the difference between K_1 and K is a single ω-triangle T and γ is a side of T;*

(c) $\operatorname{Area}(T) \leqslant \frac{1}{2}\delta(C+2)^{4m}$.

Remark. A result that is close in content was proved in [2]. A characteristic feature of our result is the additional condition that γ be one of the sides of the supplementary ω-triangle T. We use this condition essentially in what follows.

Proof. The connection γ is not $(l, \delta, C, d(K))$-insulated relative to v within $\widetilde{K}$, therefore there exists a saddle connection $\widetilde{\gamma}$ lying in $\widetilde{K}$, intersecting γ, and satisfying the conditions

$$v(\widetilde{\gamma}) \leqslant C(C+2)^{d(K)} l \quad \text{and} \quad v_\perp(\widetilde{\gamma}) \leqslant C(C+2)^{d(K)}\, \delta/l.$$

The connection γ is not a cut in K, therefore we can choose a segment AB of $\widetilde{\gamma}$ lying outside K such that A is an interior point of γ and B is either a singular point or an interior point of another saddle connection γ_1 lying at the boundary of K.

First, we shall find a segment AB_0 lying outside K, but inside $\widetilde{K}$, such that B_0 is a singular point, the interior points of AB_0 are non-singular and, in addition,

$$v(AB_0) \leqslant (C+1)(C+2)^{d(K)} l \quad \text{and} \quad v_\perp(AB_0) \leqslant (C+1)(C+2)^{d(K)}\, \delta/l.$$

If B is a singular point, then there is nothing to look for since $v(AB) \leqslant v(\widetilde{\gamma})$ and $v_\perp(AB) \leqslant v_\perp(\widetilde{\gamma})$. On the other hand, if B is an interior point of a saddle connection γ_1 at the boundary of K, then we consider two points E_1 and E_2 on this saddle connection lying on different sides of B. If E_1 and E_2 are close to B, then we can draw segments AE_1 and AE_2 that, together with AB and a piece of γ_1, bound triangles in M containing no singular points. If we start moving a point E_i ($i = 1$ or 2) away from B, then AE_i hits one or several singular points at some moment. Let B_i be the one that is the closest to A. Then the segment AB_i is a subsegment of AE_i, while the latter, together with AB and a piece of γ_1, bounds a triangle, therefore

$$v(AB_i) \leqslant v(AB) + v(\gamma_1) \leqslant C(C+2)^{d(K)} l + (C+2)^{d(K)} l = (C+1)(C+2)^{d(K)} l$$

and, in a similar way, $v_\perp(AB_i) \leqslant (C+1)(C+2)^{d(K)}\, \delta/l$. By construction, the segment AB_i lies in the complex $\widetilde{K}$ and its interior points are non-singular, that is, it is the required segment provided it is not a piece of γ. In any case, the latter is impossible for AB_1 and AB_2 simultaneously, so that the required segment AB_0 does exist.

Further, we consider a point B' on the segment AB_0. If B' is sufficiently close to A, then we can join it with the end-points of γ by segments $B'F_1$ and $B'F_2$ such that $B'F_1$, $B'F_2$, and γ bound a triangle T' containing the segment AB' but with no singular points in its interior. We now move a point B' as far from A as possible, subject to such a triangle T' still existing. For this choice of B', either some singular point is an interior point of $B'F_1$ or $B'F_2$, or $B' = B_0$ and the segments $B'F_1$ and $B'F_2$ are saddle connections. In any case, T' contains an ω-triangle T such that γ is one of its sides.

By construction, T lies outside K, but inside $\widetilde{K}$. We add T to K and obtain a new complex $K_1 \subset \widetilde{K}$ (we regard the points of γ as interior points of K_1). We claim that this is the required complex. Let γ' be a side of T different from γ. Since γ'

lies in the interior of T', its projection to the direction of γ parallel to the segment AB_0 is not longer than γ; in a similar way, the projection of γ' to the direction of AB_0 parallel to γ is not longer than AB_0. Hence

$$v(\gamma') \leqslant v(AB_0) + v(\gamma) \leqslant (C+1)(C+2)^{d(K)}l + (C+2)^{d(K)}l = (C+2)^{d(K)+1}l$$

and, in a similar way, $v_\perp(\gamma') \leqslant (C+2)^{d(K)+1}\delta/l$.

Each saddle connection at the boundary of K_1 either lies also at the boundary of K or is a side of T. Thus, bearing in mind the equality $d(K_1) = d(K)+1$, we have shown that K_1 is (l,δ,C)-close to v. In addition, it follows from the above estimates that

$$\operatorname{Area}(T) \leqslant \frac{1}{2}v(T)\cdot v_\perp(T) \leqslant \frac{1}{2}(C+2)^{d(K_1)}l\cdot(C+2)^{d(K_1)}\delta/l \leqslant \frac{1}{2}\delta(C+2)^{4m},$$

as required.

For each $C>0$ we now set $l_{\min} = l_{\min}(C) = \dfrac{s}{\sqrt{2}\,(C+2)^{2m}}$.

Lemma 4.4. *Let γ be a saddle connection (l,δ,C,d)-close to a direction v. If $l \geqslant l_{\min}(C)$ and $\delta \leqslant l^2_{\min}(C)$, then*

(a) *$v(\gamma) \geqslant v_\perp(\gamma)$ and $|\gamma| \leqslant \sqrt{2}\,v(\gamma)$;*

(b) $\dfrac{v(\gamma)}{(C+2)^{2m}} \geqslant l_{\min}$.

Proof. We have $v_\perp(\gamma) \leqslant (C+2)^d\delta/l \leqslant (C+2)^{2m}\dfrac{l^2_{\min}}{l_{\min}} = \dfrac{s}{\sqrt{2}}$. Consequently, $|\gamma| \geqslant s \geqslant \sqrt{2}\,v_\perp(\gamma)$ and we obtain the inequalities $v(\gamma) \geqslant v_\perp(\gamma)$ and $|\gamma| \leqslant \sqrt{2}\,v(\gamma)$; moreover, $v(\gamma) \geqslant \dfrac{|\gamma|}{\sqrt{2}} \geqslant \dfrac{s}{\sqrt{2}} = (C+2)^{2m}l_{\min}$.

Proposition 4.5. *Let K be a complex that is not a saddle connection, let γ be a saddle connection lying in K, let v be its direction, and let C and δ be positive constants.*

If $\delta \leqslant l^2_{\min}(C)$ and $\delta \leqslant \dfrac{\operatorname{Area}(K)}{2m(C+2)^{4m}}$, then there exist pairwise disjoint saddle connections $\gamma_0 = \gamma, \gamma_1, \dots, \gamma_n$ $(n>0)$ and a complex $\widetilde{K} \subset K$ such that the following conditions hold for some sequences of integers $d_0, d_1, \dots, d_n$ and real numbers $l_0, l_1, \dots, l_n$:

(1) $l_0 = |\gamma| \geqslant l_1 \geqslant \dots \geqslant l_n \geqslant l_{\min}$; $d_0 = 0 < d_1 < \dots < d_n \leqslant d(\widetilde{K}) < d(K)$;

(2) *$v(\gamma_i) = (C+2)^{d_i}l_i$; moreover, the saddle connection γ_i lies at the boundary of some complex K_i such that K_i is (l_i,δ,C)-close to v, $d(K_i) = d_i$, and $\gamma_0 \subset K_i \subset K$;*

(3) *the saddle connections γ_i and γ_{i+1} are sides of the same ω-triangle lying in $\widetilde{K}$;*

(4) *the saddle connection γ_n is (l_n,δ,C,d_n)-insulated relative to the direction v within K;*

(5) *the complex $\widetilde{K}$ is (l_n,δ,C)-insulated relative to v within K;*

(6) $\operatorname{Area}(\widetilde{K}) \leqslant \frac{1}{2}\,2m(C+2)^{4m}\delta \leqslant \frac{1}{2}\operatorname{Area}(K)$.

Erg

Proof. We now consider vario
and complexes $K_0 = \gamma \subset K_1$

(A1) the complex K_i has
$v(K_i) = v(\gamma_i)$;

(A2) $l_0 = |\gamma| \geqslant l_1 \geqslant \dots \geqslant l_t$

(A3) K_i is (l_i,δ,C)-close to
(l_i,δ,C)-close to v;

(A4) the difference between
γ_i is a side of T_i and

There exists at least one su
all these pairs we now choos
since $t = d(K_t) \leqslant 2m$). We cl
additional condition:

(A5) the saddle connection
within K.

For otherwise we can use L
K_{t+1} is (l_t,δ,C)-close to v, co
is a single ω-triangle T_t such
We now erase all the cuts in
of these cuts to the interior
$\operatorname{Area}(K_{t+1}) \leqslant (t+1)\,\frac{1}{2}\delta(C+$
by our choice of δ. In particul
be a saddle connection at the
set $l_{t+1} = \dfrac{v(\gamma_{t+1})}{(C+2)^{t+1}}$. The co
fore $l_{t+1} \leqslant l_t$; in addition, K
by Lemma 4.4. Hence the
$K_0, K_1, \dots, K_{t+1}$ of complexe
choice of t.

Based on (A4), we can no
that $\gamma_{t_0} = \gamma$ and the saddle
from γ_{t_i} for each $0 \leqslant i \leqslant n$
$\gamma_{t_0} = \gamma, \gamma_{t_1}, \dots, \gamma_{t_n}$ are all dis
$K_0, K_1, \dots, K_t$ lie in one ano

Continuing our constructi
$\widetilde{K}_0 = K_t \subset \widetilde{K}_1 \subset \dots \subset \widetilde{K}_u \subset$

(B1) $\widetilde{K}_i$ has no cuts and is

(B2) the difference of $\widetilde{K}_{i+1}$

Such sequences do exist
element). We now consider a
additional property:

(B3) the complex $\widetilde{K}_u$ is $(l_t$

Proof. We now consider various sequences of saddle connections $\gamma_0 = \gamma, \gamma_1, \dots, \gamma_t$ and complexes $K_0 = \gamma \subset K_1 \subset \cdots \subset K_t \subset K$ satisfying the following conditions:

(A1) the complex K_i has no cuts, γ_i lies at the boundary of K_i, and $v(K_i) = v(\gamma_i)$;

(A2) $l_0 = |\gamma| \geqslant l_1 \geqslant \cdots \geqslant l_t \geqslant l_{\min}$, where $l_i = \dfrac{v(\gamma_i)}{(C+2)^{d(K_i)}}$, $0 \leqslant i \leqslant t$;

(A3) K_i is (l_i, δ, C)-close to the direction v; for $i < t$ the complex K_{i+1} is also (l_i, δ, C)-close to v;

(A4) the difference between K_{i+1} and K_i is a single ω-triangle T_i; moreover, γ_i is a side of T_i and $\operatorname{Area}(T_i) \leqslant \frac{1}{2}\delta(C+2)^{4m}$.

There exists at least one such pair of sequences (for instance, for $t = 0$). Among all these pairs we now choose one with the largest value of t (such a pair exists since $t = d(K_t) \leqslant 2m$). We claim that this pair of sequences satisfies the following additional condition:

(A5) the saddle connection γ_t is (l_t, δ, C, t)-insulated relative to the direction v within K.

For otherwise we can use Lemma 4.3 to construct a complex $K_{t+1} \subset K$ such that K_{t+1} is (l_t, δ, C)-close to v, contains K_t, and the difference of these two complexes is a single ω-triangle T_t such that γ_t is a side of T_t and $\operatorname{Area}(T_t) \leqslant \frac{1}{2}\delta(C+2)^{4m}$. We now erase all the cuts in K_{t+1} if there are any (that is, we add the points of these cuts to the interior part of the complex). By construction, we have $\operatorname{Area}(K_{t+1}) \leqslant (t+1)\,\frac{1}{2}\delta(C+2)^{4m} \leqslant \frac{1}{2}\,2m(C+2)^{4m}\delta$, which is at most $\frac{1}{2}\operatorname{Area}(K)$ by our choice of δ. In particular, the boundary of K_{t+1} is still non-empty. Let γ_{t+1} be a saddle connection at the boundary of K_{t+1} such that $v(\gamma_{t+1}) = v(K_{t+1})$. We set $l_{t+1} = \dfrac{v(\gamma_{t+1})}{(C+2)^{t+1}}$. The complex K_{t+1} is (l_t, δ, C)-close to the direction v, therefore $l_{t+1} \leqslant l_t$; in addition, K_{t+1} is also (l_{t+1}, δ, C)-close to v. Finally, $l_{t+1} \geqslant l_{\min}$ by Lemma 4.4. Hence the sequences $\gamma_0, \gamma_1, \dots, \gamma_{t+1}$ of saddle connections and $K_0, K_1, \dots, K_{t+1}$ of complexes satisfy conditions (A1)–(A4), which contradicts our choice of t.

Based on (A4), we can now choose the numbers $t_0 < t_1 < \cdots < t_n = t$ such that $\gamma_{t_0} = \gamma$ and the saddle connection $\gamma_{t_{i+1}}$ is a side of the triangle T_{t_i} distinct from γ_{t_i} for each $0 \leqslant i \leqslant n-1$. Without loss of generality we can assume that $\gamma_{t_0} = \gamma, \gamma_{t_1}, \dots, \gamma_{t_n}$ are all distinct. Then by condition (A1) and since the complexes $K_0, K_1, \dots, K_t$ lie in one another, these saddle connections are pairwise disjoint.

Continuing our construction we consider sequences of complexes $\widetilde{K}_0, \dots, \widetilde{K}_u$, $\widetilde{K}_0 = K_t \subset \widetilde{K}_1 \subset \cdots \subset \widetilde{K}_u \subset K$, such that

(B1) $\widetilde{K}_i$ has no cuts and is (l_t, δ, C)-close to the direction v;

(B2) the difference of $\widetilde{K}_{i+1}$ and $\widetilde{K}_i$ is a single triangle of area at most $\frac{1}{2}\delta(C+2)^{4m}$.

Such sequences do exist (for instance, ones with $u = 0$, containing a single element). We now consider a sequence of largest length. This sequence satisfies an additional property:

(B3) the complex $\widetilde{K}_u$ is (l_t, δ, C)-insulated relative to the direction v within K.

This can be proved in a similar way to property (A5) above. By construction, $d(\widetilde{K}_u) = t + u \geqslant t$. Further,

$$\operatorname{Area}(\widetilde{K}_u) \leqslant d(\widetilde{K}_u) \cdot \frac{1}{2}\delta(C+2)^{4m} \leqslant 2m \cdot \frac{1}{2}\delta(C+2)^{4m} \leqslant \frac{1}{2}\operatorname{Area}(K),$$

therefore, in particular, $d(\widetilde{K}_u) < d(K)$.

Summing up, we obtain that the saddle connections $\gamma, \gamma_{t_1}, \dots, \gamma_{t_n}$ and the complex $\widetilde{K}_u$ satisfy conditions (1)–(6) for the sequences of indices $0, t_1, \dots, t_n$ and real numbers $l_0, l_1, \dots, l_n$.

Let K be some complex and let Γ be a saddle connection at its boundary. For each $L > 0$ let $N_1(K, \Gamma; L)$ be the number of saddle connections γ such that

(S1) $|\gamma| \leqslant L\,|\Gamma|$;

(S2) γ and Γ are sides of some ω-triangle T_γ lying in K.

Let $N_2(K, \Gamma; L)$ be the number of saddle connections γ satisfying the additional two conditions

(S3) the angle between γ and Γ in T_γ is acute and

(S4) $|\gamma| \geqslant |\Gamma|/2$.

An intermediate step in our proof of Theorem 4.1 is an estimate of the value of $N_1(K, \Gamma; L)$, which we now embark on. The final product here is Theorem 4.15.

Lemma 4.6. *We have $N_1(K, \Gamma; L) \leqslant 3 \cdot N_2(K, \Gamma; L+1)$ for each $L > 0$.*

Proof. For each saddle connection γ satisfying (S1) and (S2) let $\widetilde{\gamma}$ be the side of T_γ distinct from γ and Γ. If γ fails to satisfy one of conditions (S3) and (S4), then, clearly, $\widetilde{\gamma}$ satisfies both. In addition, $\widetilde{\gamma}$ satisfies (S2) and $|\widetilde{\gamma}| \leqslant |\gamma| + |\Gamma| \leqslant (L+1)|\Gamma|$. We note further that an ω-triangle with sides $\widetilde{\gamma}$ and Γ is unambiguously defined if one indicates on which side of either saddle connection it lies. Thus, there are at most four such triangles and it is easy to see that there are at most two of them with an acute angle between $\widetilde{\gamma}$ and Γ. Hence an arbitrary saddle connection can play the role of $\widetilde{\gamma}$ for at most two saddle connections γ failing to satisfy (S3) or (S4), which gives the required estimate.

Let C, δ, and L be positive numbers and let n, n_1, and n_2 be integers, $0 \leqslant n, n_1, n_2 < 2m$. We say that pairwise disjoint saddle connections $\gamma_0, \dots, \gamma_n$ and a complex $\widetilde{K} \subset K$ *have the property* $P_3 = P_3(K, \Gamma; C, \delta, L; n, n_1, n_2)$ if there exists a direction v and sequences $d_0, \dots, d_n$ of integers and $l_0, \dots, l_n$ of real numbers such that

(a) conditions (1)–(6) in Proposition 4.5 are satisfied;

(b) $d_n = n_1$ and $d(\widetilde{K}) = n_2$;

(c) γ_0 has properties (S1)–(S4).

We denote the number of all such collections by $N_3(K, \Gamma; C, \delta, L; n, n_1, n_2)$.

Lemma 4.7. *If $\delta \leqslant l^2_{\min}(C)$ and $\delta \leqslant \dfrac{\operatorname{Area}(K)}{2m(C+2)^{4m}}$, then*

$$N_2(K, \Gamma; L) \leqslant \sum_{0 \leqslant n, n_1, n_2 < d(K)} N_3(K, \Gamma; C, \delta, L; n, n_1, n_2) \quad \textit{for each} \quad L > 0.$$

The proof immediately follows from Proposition 4.5.

Further, let $\widetilde{L}$ and σ be positive numbers, $0 < \sigma < 1$, and let $k_0, \dots, k_n$ be non-negative integers. We can define the property

$$\widetilde{P}_3 = \widetilde{P}_3(K, \Gamma; C, \delta, L, \sigma, \widetilde{L}; n, n_1, n_2; k_0, \dots, k_n)$$

using the definition of P_3 with (c) replaced by the following two conditions:

(c′) γ_0 has properties (S1) and (S2);

(d′) $\sigma L_i |\Gamma| \leqslant v(\gamma_i) \leqslant L_i |\Gamma|$, where $L_i = \sigma^{k_i} \widetilde{L}$, $0 \leqslant i \leqslant n$.

We denote the number of collections with property $\widetilde{P}_3$ by

$$\widetilde{N}_3(K, \Gamma; C, \delta, L, \sigma, \widetilde{L}; n, n_1, n_2; k_0, \dots, k_n).$$

Lemma 4.8. *Assume that saddle connections $\gamma_0, \gamma_1, \dots, \gamma_n$ and a complex $\widetilde{K}$ satisfy the property $P_3(K, \Gamma; C, \delta, L; n, n_1, n_2)$. If $\delta \leqslant l^2_{\min}(C)$, then*

$$\frac{1}{4}|\Gamma| \leqslant v(\gamma_i) \leqslant (C+2)^{n_1} L |\Gamma|, \qquad 0 \leqslant i \leqslant n.$$

Proof. The upper estimate is obvious:

$$v(\gamma_i) = (C+2)^{d_i} l_i \leqslant (C+2)^{n_1} |\gamma_0| \leqslant (C+2)^{n_1} L |\Gamma|.$$

Further, by properties (S2)–(S4), the saddle connections γ_0 and Γ are sides of some ω-triangle $T \subset K$, the angle between them in this triangle is acute, and $|\gamma_0| \geqslant |\Gamma|/2$. By condition (2) in Proposition 4.5 the saddle connection γ_i lies at the boundary of some complex K_i, $\gamma_0 \subset K_i \subset K$. If one of the saddle connections γ_0 and Γ lies at the boundary of K_i, then $|K_i| \geqslant \frac{1}{2}|\Gamma|$. Otherwise γ_0 lies in the interior of K_i and Γ lies outside K_i. Hence there exists a saddle connection $\widetilde{\gamma}$ at the boundary of K_i that starts at the common vertex of γ_0 and Γ and intersects T. Since the angle between γ_0 and Γ is acute, the length of $\widetilde{\gamma}$ is at least $\frac{1}{\sqrt{2}} \min(|\gamma_0|, |\Gamma|)$, which is greater than or equal to $\frac{1}{2\sqrt{2}}|\Gamma|$.

Thus in any case $|K_i| \geqslant \frac{1}{2\sqrt{2}}|\Gamma|$. By the same condition (2) the complex K_i is (l_i, δ, C)-close to the direction v, therefore $v(K_i) \geqslant \frac{1}{\sqrt{2}}|K_i| \geqslant \frac{1}{4}|\Gamma|$ by Lemma 4.4. Finally, $v(\gamma_i) = v(K_i)$.

Corollary 4.9. *For $\delta \leqslant l^2_{\min}(C)$ we have*

$$N_3(K, \Gamma; C, \delta, L; n, n_1, n_2) \leqslant \sum_{0 \leqslant k_0, \dots, k_n \leqslant k(L, \sigma, C; n_1)} \widetilde{N}_3(K, \Gamma; C, \delta, L, \sigma, (C+2)^{n_1} L; n, n_1, n_2; k_0, \dots, k_n),$$

where $k(L, \sigma, C; n_1) = \log_{1/\sigma}(4\,(C+2)^{n_1} L)$.

Let $S_4(K, \Gamma; C, \delta, \sigma, L; n, n_1)$ be the set of saddle connections occurring as γ_n in various collections $\gamma_0, \dots, \gamma_n, \widetilde{K}$ satisfying the property

$$\widetilde{P}_3 = \widetilde{P}_3(K, \Gamma; C, \delta, L', \sigma, \widetilde{L}; n, n_1, n_2; k_0, \dots, k_n)$$

for some values of L', $\widetilde{L}$, n_2, and $k_0, \dots, k_n$ such that $\sigma^{k_n} \widetilde{L} = L$. We denote the number of elements in this set by $N_4(K, \Gamma; C, \delta, \sigma, L; n, n_1)$.

Lemma 4.10. *Let $\gamma, \widetilde{\gamma} \in S_4(K, \Gamma; C, \delta, \sigma, L; n, n_1)$ be intersecting saddle connections. If $\delta \leqslant l_{\min}^2(C)$ and $C\sigma \geqslant \sqrt{2}$, then*

$$\angle(\gamma, \widetilde{\gamma}) > \delta \left(\frac{C}{\sqrt{2}} - 1 \right) \frac{(C+2)^{2n_1}}{L^2 |\Gamma|^2}.$$

Remark. Here and in what follows, we choose the value of the angle between two saddle connections or between a saddle connection and a direction in the range from 0 to $\pi/2$.

Proof of Lemma 4.10. By the definition of the set S_4, the saddle connection γ is (l, δ, C, n_1)-insulated and $\widetilde{\gamma}$ is $(\widetilde{l}, \delta, C, n_1)$-insulated within K relative to some directions v and $\widetilde{v}$, respectively, where $v(\gamma) = (C+2)^{n_1} l$, $\widetilde{v}(\widetilde{\gamma}) = (C+2)^{n_1} \widetilde{l}$, $\sigma L |\Gamma| \leqslant v(\gamma), \widetilde{v}(\widetilde{\gamma}) \leqslant L |\Gamma|$, $l \geqslant l_{\min}(C)$, and $\widetilde{l} \geqslant l_{\min}(C)$.

For definiteness, assume that $l \geqslant \widetilde{l}$. Since γ is (l, δ, C, n_1)-close to v, it follows that $v_\perp(\gamma) \leqslant (C+2)^{n_1} \delta / l$. Hence

$$\angle(\gamma, v) \leqslant \tan \angle(\gamma, v) = \frac{v_\perp(\gamma)}{v(\gamma)} \leqslant \frac{(C+2)^{n_1} \delta / l}{(C+2)^{n_1} l} = \frac{\delta}{l^2}.$$

Further, $\widetilde{v}(\widetilde{\gamma}) \leqslant \frac{1}{\sigma} v(\gamma)$ and, in view of Lemma 4.4, we obtain $|\widetilde{\gamma}| \leqslant \sqrt{2}\, \widetilde{v}(\widetilde{\gamma})$; hence

$$v(\widetilde{\gamma}) \leqslant |\widetilde{\gamma}| \leqslant \sqrt{2}\, \widetilde{v}(\widetilde{\gamma}) \leqslant \frac{\sqrt{2}}{\sigma} v(\gamma).$$

Since $C\sigma \geqslant \sqrt{2}$, it follows that $v(\widetilde{\gamma}) \leqslant C\, v(\gamma) = C(C+2)^{n_1} l$. The saddle connection γ is insulated relative to v, therefore $v_\perp(\widetilde{\gamma}) > C(C+2)^{n_1} \delta / l$ by this estimate and

$$\angle(\widetilde{\gamma}, v) \geqslant \sin \angle(\widetilde{\gamma}, v) = \frac{v_\perp(\widetilde{\gamma})}{|\widetilde{\gamma}|} > \frac{C(C+2)^{n_1} \delta / l}{\sqrt{2}\, \widetilde{v}(\widetilde{\gamma})} = \frac{C}{\sqrt{2}} \cdot \frac{\delta}{l \widetilde{l}} \geqslant \frac{C}{\sqrt{2}} \cdot \frac{\delta}{l^2}.$$

It remains to observe that $\angle(\gamma, \widetilde{\gamma}) \geqslant \angle(\widetilde{\gamma}, v) - \angle(\gamma, v)$, while by the above,

$$\angle(\widetilde{\gamma}, v) - \angle(\gamma, v) > \left(\frac{C}{\sqrt{2}} - 1 \right) \frac{\delta}{l^2} = \delta \left(\frac{C}{\sqrt{2}} - 1 \right) \frac{(C+2)^{2n_1}}{(v(\gamma))^2} \geqslant \delta \left(\frac{C}{\sqrt{2}} - 1 \right) \frac{(C+2)^{2n_1}}{L^2 |\Gamma|^2}.$$

Lemma 4.11. *If $\delta \leqslant l_{\min}^2(C)$ and $C\sigma \geqslant \sqrt{2}$, then*

$$N_4(K, \Gamma; C, \delta, \sigma, L; n, n_1) \leqslant 3m \left(\frac{2\pi \cdot \mathrm{Area}(K) \cdot 1/\sigma}{\delta(\frac{C}{\sqrt{2}} - 1)(C+2)^{n_1}} L + \frac{2(1/\sigma)^2}{\frac{C}{\sqrt{2}} - 1} + 1 \right).$$

Proof. Let γ be a saddle connection in the set $S_4(K, \Gamma; C, \delta, \sigma, L; n, n_1)$ and let $\gamma_0, \ldots, \gamma_n = \gamma$ and $\widetilde{K}$ be the corresponding collection satisfying the property $\widetilde{P}_3$.

Let v be the direction of γ_0
some triangle T lying in K,
$\sin \angle(\Gamma$
Hence
$\angle(\Gamma, v) = \angle$
By property $\widetilde{P}_3$ we have σL
$\angle($
Next,
$\angle(\gamma, v) \leqslant$ tan
where $v(\gamma) = (C+2)^{n_1} l$. U
Thus,
$\angle(\Gamma, \gamma) \leqslant \angle(\Gamma, v) +$
We denote the right-hand s
directions of all the saddle c
We partition this arc into s
and maybe also a subarc o
is at most $\frac{2\varphi}{\varphi_0} + 1$. By L
belonging to the same sub
Proposition 4.2. Consequen
$N_4(K,$
which, on substituting our
lemma.
Let γ be an arbitrary sa
complex $\widetilde{K} \subset K$ *has prope*
a real number $l \geqslant l_{\min}(C)$
(a) γ is (l, δ, C, n_1)-clos
(b) $\widetilde{K}$ is (l, δ, C)-insula
Let $N_5(K, \gamma; C, \delta; n_1, n$
We denote the set of
$\widetilde{S}_5(K, \gamma; C, \delta; n_1, n_2)$.

Let v be the direction of γ_0. Then the saddle connections γ_0 and Γ are sides of some triangle T lying in K, therefore

$$\sin\angle(\Gamma,\gamma_0)=\frac{2\operatorname{Area}(T)}{|\Gamma|\,|\gamma_0|}\leqslant\frac{2\operatorname{Area}(K)}{|\Gamma|\,|\gamma_0|}.$$

Hence

$$\angle(\Gamma,v)=\angle(\Gamma,\gamma_0)\leqslant\frac{\pi}{2}\sin\angle(\Gamma,\gamma_0)\leqslant\frac{\pi\operatorname{Area}(K)}{|\Gamma|\,|\gamma_0|}.$$

By property $\widetilde{P}_3$ we have $\sigma L|\Gamma|\leqslant v(\gamma)\leqslant L|\Gamma|$ and $v(\gamma)\leqslant(C+2)^{n_1}|\gamma_0|$, therefore

$$\angle(\Gamma,v)\leqslant\frac{\pi\operatorname{Area}(K)}{\sigma L|\Gamma|^2}(C+2)^{n_1}.$$

Next,

$$\angle(\gamma,v)\leqslant\tan\angle(\gamma,v)=\frac{v_\perp(\gamma)}{v(\gamma)}\leqslant\frac{(C+2)^{n_1}\delta/l}{(C+2)^{n_1}l}=\frac{\delta}{l^2},$$

where $v(\gamma)=(C+2)^{n_1}l$. Using the inequality $\sigma L\,|\Gamma|\leqslant v(\gamma)$ again we obtain

$$\angle(\gamma,v)\leqslant\frac{\delta(C+2)^{2n_1}}{\sigma^2L^2|\Gamma|^2}.$$

Thus,

$$\angle(\Gamma,\gamma)\leqslant\angle(\Gamma,v)+\angle(\gamma,v)\leqslant\frac{\pi\operatorname{Area}(K)}{\sigma L|\Gamma|^2}(C+2)^{n_1}+\frac{\delta(C+2)^{2n_1}}{\sigma^2L^2|\Gamma|^2}.$$

We denote the right-hand side of the resulting inequality by φ. By the above, the directions of all the saddle connections in the set S_4 belong to some arc of length 2φ. We partition this arc into subarcs of length

$$\varphi_0=\delta\left(\frac{C}{\sqrt{2}}-1\right)\frac{(C+2)^{2n_1}}{L^2|\Gamma|^2}$$

and maybe also a subarc of length smaller than φ_0. The number of these subarcs is at most $\dfrac{2\varphi}{\varphi_0}+1$. By Lemma 4.10, saddle connections in S_4 with directions belonging to the same subarc are disjoint; hence their number is at most $3m$ by Proposition 4.2. Consequently,

$$N_4(K,\Gamma;C,\delta,\sigma,L;n,n_1)\leqslant 3m\left(\frac{2\varphi}{\varphi_0}+1\right),$$

which, on substituting our expressions for φ and φ_0, delivers the assertion of the lemma.

Let γ be an arbitrary saddle connection and let K be a complex. We say that a complex $\widetilde{K}\subset K$ *has property* $P_5(K,\gamma;C,\delta;n_1,n_2)$ if there exist a direction v and a real number $l\geqslant l_{\min}(C)$ such that

(a) γ is (l,δ,C,n_1)-close to v and $v(\gamma)=(C+2)^{n_1}l$;

(b) $\widetilde{K}$ is (l,δ,C)-insulated relative to v within K and $d(\widetilde{K})=n_2$.

Let $N_5(K,\gamma;C,\delta;n_1,n_2)$ be the number of complexes with property P_5. We denote the set of saddle connections bounding these complexes by $\widetilde{S}_5(K,\gamma;C,\delta;n_1,n_2)$.

Lemma 4.12. *If $\delta \leqslant l_{\min}^2(C)$ and $C \geqslant 4\sqrt{2}$, then the set $\widetilde{S}_5(K,\gamma;C,\delta;n_1,n_2)$ consists of pairwise disjoint saddle connections.*

Proof. Assume the contrary: let γ' and γ'' be intersecting saddle connections in $\widetilde{S}_5(K,\gamma;C,\delta;n_1,n_2)$. Let K' and K'' be complexes satisfying property P_5 such that γ' and γ'', respectively, lie at their boundaries. Let v', v'' and l', l'' be the directions and real numbers corresponding to K' and K''. By condition (a) in our definition of property P_5,

$$v'(\gamma) = (C+2)^{n_1} l', \qquad v'_\perp(\gamma) \leqslant (C+2)^{n_1}\delta/l',$$
$$v''(\gamma) = (C+2)^{n_1} l'', \qquad v''_\perp(\gamma) \leqslant (C+2)^{n_1}\delta/l'',$$

therefore

$$\angle(v',\gamma) \leqslant \tan\angle(v',\gamma) = \frac{v'_\perp(\gamma)}{v'(\gamma)} \leqslant \frac{(C+2)^{n_1}\delta/l'}{(C+2)^{n_1} l'} = \frac{\delta}{(l')^2},$$

and in the same way,

$$\angle(v'',\gamma) \leqslant \frac{\delta}{(l'')^2}.$$

Hence

$$\angle(v',v'') \leqslant \angle(v',\gamma) + \angle(v'',\gamma) \leqslant \frac{\delta}{(l')^2} + \frac{\delta}{(l'')^2}.$$

It follows by Lemma 4.4 that $\frac{1}{\sqrt{2}}|\gamma| \leqslant v'(\gamma), v''(\gamma) \leqslant |\gamma|$. In particular, we have $\frac{1}{\sqrt{2}} \leqslant l'/l'' \leqslant \sqrt{2}$, therefore the angle $\angle(v',v'')$ is not larger than $3\dfrac{\delta}{(l')^2}$ or $3\dfrac{\delta}{(l'')^2}$.

We now use condition (b) in the definition of P_5. For definiteness, assume that $l' \geqslant l''$. The saddle connection γ' lies at the boundary of K', therefore we have $v'(\gamma') \leqslant (C+2)^{n_2} l'$ and $v'_\perp(\gamma') \leqslant (C+2)^{n_2}\delta/l'$, so that

$$\angle(v',\gamma') \leqslant \tan\angle(v',\gamma') \leqslant \frac{(C+2)^{n_2}\delta/l'}{v'(\gamma')}.$$

On the other hand γ' intersects the boundary of K'', therefore $v''(\gamma') > C(C+2)^{n_2} l''$ or $v''_\perp(\gamma') > C(C+2)^{n_2}\delta/l''$. The first of these two inequalities fails, for

$$v''(\gamma') \leqslant |\gamma'| \leqslant \sqrt{2}\, v'(\gamma') \leqslant \sqrt{2}(C+2)^{n_2} l' \leqslant 2(C+2)^{n_2} l''$$

by Lemma 4.4, while $C \geqslant 2$. Hence we have the second inequality, from which it follows that

$$\angle(v'',\gamma') \geqslant \sin\angle(v'',\gamma') > \frac{C(C+2)^{n_2}\delta/l''}{|\gamma'|} \geqslant \frac{C(C+2)^{n_2}\delta/l'}{\sqrt{2}\, v'(\gamma')}.$$

As a result,

$$\angle(v',v'') \geqslant \angle(v'',\gamma') - \angle(v',\gamma') > \left(\frac{C}{\sqrt{2}} - 1\right)\frac{(C+2)^{n_2}\delta/l'}{v'(\gamma')} \geqslant \left(\frac{C}{\sqrt{2}} - 1\right)\frac{\delta}{(l')^2},$$

which for $C \geqslant 4\sqrt{2}$ contradicts our earlier estimate $\angle(v',v'') \leqslant 3\dfrac{\delta}{(l')^2}$.

Corollary 4.13. *If $\delta \leqslant l^2_{\mathrm{m}}$*

Proof. Since the saddle co
there are at most $3m$ of th
ing an arbitrary complex
therefore there are at most
of saddle connections can

Let K be a complex t
connection at the bounda
property: for each complex
relations $0 < d(\widetilde{K}) < d(K$
have

Lemma 4.14. *If $\delta \leqslant l^2_{\min}$*

$\widetilde{N}_3(K,\Gamma;C,\delta,L,\sigma,$

$\leqslant N_4(K$

where $L_i = \sigma^{k_i}\widetilde{L}$, $0 \leqslant i \leqslant$

Proof. We must find an e
that

$\widetilde{P}_3\,(K,$

The number of saddle con
most $N_4(K,\Gamma;C,\delta,\sigma,L_n;$
play the role of $\widetilde{K}$ is at m
by Corollary 4.13. The p
show that, for fixed saddl
of saddle connections that
most $\widetilde{N}\left(\dfrac{\sqrt{2}}{\sigma}\dfrac{L_i}{L_{i+1}}\right)$ eleme

We now add the saddle
that it does not already lie
by $\widetilde{K}_{i+1}$. This complex
$0 < d(\widetilde{K}_{i+1}) = d(\widetilde{K}) < d($
Let γ be a saddle connec
γ_{i+1} are two sides of th

Corollary 4.13. *If $\delta \leqslant l^2_{\min}(C)$ and $C \geqslant 4\sqrt{2}$, then*

$$N_5(K,\gamma;C,\delta;n_1,n_2) \leqslant 2^{3m+1}.$$

Proof. Since the saddle connections in $\widetilde{S}_5(K,\gamma;C,\delta;n_1,n_2)$ are pairwise disjoint, there are at most $3m$ of them by Proposition 4.2. Further, the connections bounding an arbitrary complex with property P_5 form a subset of $\widetilde{S}_5(K,\gamma;C,\delta;n_1,n_2)$, therefore there are at most 2^{3m} various boundaries. Finally, an arbitrary collection of saddle connections can be the boundary of at most two complexes.

Let K be a complex that is not a saddle connection and let Γ be a saddle connection at the boundary of K. Let $\widetilde{N}(L)$ be a function with the following property: for each complex $\widetilde{K} \subset K$ containing $\widetilde{\Gamma}$ at its boundary and satisfying the relations $0 < d(\widetilde{K}) < d(K)$ and $\operatorname{Area}(\widetilde{K}) \leqslant m(C+2)^{4m}\delta$ and for each $L > 0$ we have

$$N_1(\widetilde{K},\widetilde{\Gamma};L) \leqslant \widetilde{N}(L).$$

Lemma 4.14. *If $\delta \leqslant l^2_{\min}(C)$ and $C \geqslant 4\sqrt{2}$, then*

$$\widetilde{N}_3(K,\Gamma;C,\delta,L,\sigma,\widetilde{L};n,n_1,n_2;k_0,\dots,k_n)$$
$$\leqslant N_4(K,\Gamma;C,\delta,\sigma,L_n;n,n_1)\cdot 2^{3m+1}\cdot \prod_{i=0}^{n-1}\widetilde{N}\left(\frac{\sqrt{2}}{\sigma}\,\frac{L_i}{L_{i+1}}\right),$$

where $L_i = \sigma^{k_i}\widetilde{L}$, $0 \leqslant i \leqslant n$.

Proof. We must find an estimate for the number of collections $\gamma_0,\dots,\gamma_n,\widetilde{K}$ such that

$$\widetilde{P}_3\,(K,\Gamma;\,C,\delta,L,\sigma,\widetilde{L}\,;n,n_1,n_2;\,k_0,\dots,k_n).$$

The number of saddle connections that can occur as γ_n in such a collection is at most $N_4(K,\Gamma;C,\delta,\sigma,L_n;n,n_1)$. For γ_n fixed, the number of complexes that can play the role of $\widetilde{K}$ is at most $N_5(K,\gamma_n;C,\delta;n_1,n_2)$, which has the estimate 2^{3m+1} by Corollary 4.13. The proof is complete for $n=0$. For $n>0$ it remains to show that, for fixed saddle connections $\gamma_{i+1},\dots,\gamma_n$ and a complex $\widetilde{K}$, the set S of saddle connections that can occur as γ_i in a collection with property $\widetilde{P}_3$ has at most $\widetilde{N}\left(\frac{\sqrt{2}}{\sigma}\,\frac{L_i}{L_{i+1}}\right)$ elements.

We now add the saddle connection γ_{i+1} to the boundary of $\widetilde{K}$ as a cut (provided that it does not already lie at the boundary of $\widetilde{K}$); we denote the resulting complex by $\widetilde{K}_{i+1}$. This complex lies in K, $\operatorname{Area}(\widetilde{K}_{i+1}) = \operatorname{Area}(\widetilde{K}) \leqslant m(C+2)^{4m}\delta$, and $0 < d(\widetilde{K}_{i+1}) = d(\widetilde{K}) < d(K)$, therefore $N_1(\widetilde{K}_{i+1},\gamma_{i+1};L) \leqslant \widetilde{N}(L)$ for each $L>0$. Let γ be a saddle connection in the set S. Then the saddle connections γ and γ_{i+1} are two sides of the same ω-triangle lying in $\widetilde{K}$ and therefore in $\widetilde{K}_{i+1}$.

Further, $\sigma L_{i+1}|\Gamma| \leqslant v(\gamma_{i+1}) \leqslant L_{i+1}|\Gamma|$ and $\sigma L_i|\Gamma| \leqslant v(\gamma) \leqslant L_i|\Gamma|$ for the corresponding direction v, where $|\gamma| \leqslant \sqrt{2}\, v(\gamma)$ by Lemma 4.4. Hence

$$|\gamma| \leqslant \sqrt{2}\, v(\gamma) \leqslant \sqrt{2}\, L_i|\Gamma| = \frac{\sqrt{2}}{\sigma}\,\frac{L_i}{L_{i+1}}\,\sigma L_{i+1}|\Gamma| \leqslant \frac{\sqrt{2}}{\sigma}\,\frac{L_i}{L_{i+1}}\,v(\gamma_{i+1}) \leqslant \frac{\sqrt{2}}{\sigma}\,\frac{L_i}{L_{i+1}}\,|\gamma_{i+1}|.$$

Thus, the number of elements of S is at most

$$N_1\left(\widetilde{K}_{i+1}, \gamma_{i+1}; \frac{\sqrt{2}}{\sigma}\,\frac{L_i}{L_{i+1}}\right) \leqslant \widetilde{N}\left(\frac{\sqrt{2}}{\sigma}\,\frac{L_i}{L_{i+1}}\right).$$

We now set $\widetilde{N}_1(1; L) = A_1\, L$, while for $d > 1$ we set

$$\widetilde{N}_1(d; L) = A_d\, L\,\left(\log_4(B_d\, L)\right)^{r_d},$$

where $A_d = (3 \cdot 2^6)^{(2m)^d}$, $B_d = (2^{6m})^{d-1}$, and $r_d = (2m)^{d-1}$. We note that $2 \leqslant \widetilde{N}_1(d; L) \leqslant \widetilde{N}_1(d+1; L)$ for $L \geqslant 1$, $d = 1, 2, \dots$.

Theorem 4.15. *Let K be a complex such that $0 < d(K) < 2m$ and let Γ be a boundary saddle connection. If $\operatorname{Area}(K) \leqslant m \cdot s^2$, then*

$$N_1(K, \Gamma; L) \leqslant \widetilde{N}_1(d(K); L) \quad \textit{for } L \geqslant 1.$$

Proof. We proceed by induction on $d(K)$. If $d(K) = 1$, then K is an ω-triangle and $N_1(K, \Gamma; L) \leqslant 2$ for each $L > 0$; in particular, we obtain the assertion of the theorem.

Assume now that $d = d(K) > 1$ and that we have proved this assertion for all the complexes $\widetilde{K}$ with $d(\widetilde{K}) < d$. We set $C = 6$, $\sigma = 1/4$, and $\delta = \dfrac{\operatorname{Area}(K)}{2m \cdot 8^{4m}}$. For this choice of constants we have $\delta \leqslant \dfrac{\operatorname{Area}(K)}{2m(C+2)^{4m}}$, $\delta \leqslant l^2_{\min}(C)$, $C\sigma \geqslant \sqrt{2}$, and $C \geqslant 4\sqrt{2}$, which enables us to use all the results of this section. Further, we set $\widetilde{N}(L) = \widetilde{N}_1(d-1; L)$ for $L \geqslant 1$ and $\widetilde{N}(L) = \widetilde{N}_1(d-1; 1)$ for $L < 1$. By the induction hypothesis we can apply Lemma 4.14 to K for this choice of $\widetilde{N}(L)$.

We discuss the case of $d - 1 > 1$ first. We start with a suitable estimate of the quantity $N_4 = N_4(K, \Gamma; C, \delta, \sigma, L; n, n_1)$ for $L \geqslant 1/4$. Writing the inequality of Lemma 4.11 for the particular values of C, σ, and δ we obtain

$$N_4 \leqslant 3m\left(\frac{2\pi(2m \cdot 8^{4m}) \cdot 4}{(6/\sqrt{2}-1)8^{n_1}}\,L + \frac{32}{6/\sqrt{2}-1} + 1\right) \leqslant m\,(16\pi m \cdot 8^{4m-n_1} L + 35).$$

By the condition $L \geqslant 1/4$,

$$N_4 \leqslant m\,(16\pi m \cdot 8^{4m-n_1} + 4 \cdot 35)L = m^2 8^{4m-n_1}\left(16\pi + \frac{140}{m \cdot 8^{4m-n_1}}\right)L.$$

$| \leqslant v(\gamma) \leqslant L_i|\Gamma|$ for the cor-
ıma 4.4. Hence

$\cdot \sigma L_{i+1}|\Gamma|$

$\frac{\sqrt{2}}{\sigma}\frac{L_i}{L_{i+1}}|\gamma_{i+1}|.$

$\left(\frac{\sqrt{2}}{\sigma}\frac{L_i}{L_{i+1}}\right).$

set

$L))^{r_d},$

$r_d = (2m)^{d-1}$. We note that

$< d(K) < 2m$ *and let* Γ *be a*
then

for $L \geqslant 1$.

$) = 1$, then K is an ω-triangle
, we obtain the assertion of the

ave proved this assertion for all
$, \sigma = 1/4$, and $\delta = \frac{\text{Area}(K)}{2m \cdot 8^{4m}}$.
$\frac{K)}{\cdot 2)^{4m}}$, $\delta \leqslant l^2_{\min}(C)$, $C\sigma \geqslant \sqrt{2}$,
ılts of this section. Further, we
$\widetilde{N}_1(d-1;1)$ for $L < 1$. By the
K for this choice of $\widetilde{N}(L)$.
art with a suitable estimate of
$\geqslant 1/4$. Writing the inequality of
δ we obtain

$\leqslant m\,(16\pi m \cdot 8^{4m-n_1}L + 35).$

$^{-n_1}\left(16\pi + \frac{140}{m \cdot 8^{4m-n_1}}\right)L.$

Further, $m \geqslant 1$ and $n_1 < d(K) < 2m$, therefore

$$16\pi + \frac{140}{m \cdot 8^{4m-n_1}} \leqslant 16\pi + \frac{140}{8^4},$$

which is smaller than 51. Hence

$$N_4 \leqslant (51m^2 \cdot 8^{4m-n_1})L.$$

It now follows by Lemma 4.14 that

$$\begin{aligned}\widetilde{N}_3 &= \widetilde{N}_3\big(K,\Gamma;C,\delta,L,\sigma,(C+2)^{n_1}L;n,n_1,n_2;k_0,\dots,k_n\big)\\ &\leqslant 51m^2 \cdot 8^{4m-n_1}\,L_n \cdot 2^{3m+1} \cdot \prod_{i=0}^{n-1}\widetilde{N}\left(\frac{\sqrt{2}}{\sigma}\frac{L_i}{L_{i+1}}\right)\\ &\text{for } L_i = \sigma^{k_i}(C+2)^{n_1}L, \qquad 0 \leqslant i \leqslant n, \end{aligned} \tag{1}$$

and $0 \leqslant k_i \leqslant k(L,\sigma,C;n_1) = \log_{1/\sigma}(4(C+2)^{n_1}L)$, since $k_n \leqslant k(L,\sigma,C;n_1)$, which ensures that $L_n \geqslant 1/4$.

We now consider two cases, $n > 0$ and $n = 0$. Assume that $n > 0$. For $\frac{L_i}{L_{i+1}} \geqslant 1$ we also have $\frac{\sqrt{2}}{\sigma}\frac{L_i}{L_{i+1}} = 4\sqrt{2}\frac{L_i}{L_{i+1}} \geqslant 1$, therefore

$$\widetilde{N}\left(\frac{\sqrt{2}}{\sigma}\frac{L_i}{L_{i+1}}\right) \leqslant A_{d-1} \cdot 4\sqrt{2}\frac{L_i}{L_{i+1}} \cdot \left(\log_4\left(B_{d-1} \cdot 4\sqrt{2}\frac{L_i}{L_{i+1}}\right)\right)^{r_{d-1}}.$$

On the other hand, if $\frac{L_i}{L_{i+1}} < 1$, then $\frac{\sqrt{2}}{\sigma}\frac{L_i}{L_{i+1}} < \frac{\sqrt{2}}{\sigma} = 4\sqrt{2}$ and

$$\widetilde{N}\left(\frac{\sqrt{2}}{\sigma}\frac{L_i}{L_{i+1}}\right) \leqslant A_{d-1} \cdot 4\sqrt{2} \cdot \left(\log_4(B_{d-1} \cdot 4\sqrt{2})\right)^{r_{d-1}}.$$

Since $0 \leqslant k_i, k_{i+1} \leqslant k(L,\sigma,C;n_1)$, it follows that $\frac{L_i}{L_{i+1}} \leqslant 4(C+2)^{n_1}L$ and $1 \leqslant 4(C+2)^{n_1}L$. Hence

$$\begin{aligned}\prod_{i=0}^{n-1}\widetilde{N}\left(\frac{\sqrt{2}}{\sigma}\frac{L_i}{L_{i+1}}\right) &\leqslant (A_{d-1}\cdot 4\sqrt{2})^n\frac{L_0}{L_n}\cdot\left(\prod_{i=0}^{n-1}\max\left(1,\frac{L_{i+1}}{L_i}\right)\right)\\ &\quad\times\left(\log_4\left(B_{d-1}\cdot 4\sqrt{2}\cdot 4(C+2)^{n_1}L\right)\right)^{nr_{d-1}}.\end{aligned}$$

Assume that we have chosen $k_0,\dots,k_n$ such that $\widetilde{N}_3 \neq 0$. We establish an estimate for $\prod_{i=0}^{n-1}\max\left(1,\frac{L_{i+1}}{L_i}\right)$ in this case. We find some saddle connections $\gamma_0,\dots,\gamma_n$ and a complex $\widetilde{K}$ such that $\widetilde{P}_3$ holds with suitable parameters.

Let v be the direction of γ_0 and let $d_0 = 0 < d_1 < \cdots < d_n = n_1$ be the corresponding integers (see Proposition 4.5). Then

$$\frac{v(\gamma_{i+1})}{v(\gamma_i)} \leqslant (C+2)^{d_{i+1}-d_i},$$

and since $\sigma L_{i+1}|\Gamma| \leqslant v(\gamma_{i+1}) \leqslant L_{i+1}|\Gamma|$ and $\sigma L_i|\Gamma| \leqslant v(\gamma) \leqslant L_i|\Gamma|$, it follows that

$$\frac{L_{i+1}}{L_i} \leqslant \frac{1}{\sigma}(C+2)^{d_{i+1}-d_i}.$$

Hence

$$\prod_{i=0}^{n-1} \max\left(1, \frac{L_{i+1}}{L_i}\right) \leqslant (1/\sigma)^n (C+2)^{n_1} = 4^n \cdot 8^{n_1},$$

and

$$\prod_{i=0}^{n-1} \widetilde{N}\left(\frac{\sqrt{2}}{\sigma}\frac{L_i}{L_{i+1}}\right) \leqslant A_{d-1}^{2m-2}\,(16\sqrt{2})^{2m-2}\,8^{n_1}\,\frac{L_0}{L_n}\left(\log_4\left(B_{d-1}\,\frac{1}{2\sqrt{2}}\,8^{2m}L\right)\right)^{(2m-2)r_{d-1}}$$

for $\widetilde{N}_3 \neq 0$ (we use the fact that $n \leqslant n_1 \leqslant 2m-2$). Further, by the conditions $|\gamma_0| \leqslant L\,|\Gamma|$ and $\sigma L_0|\Gamma| \leqslant |\gamma_0| \leqslant L_0|\Gamma|$ we obtain that $L_0 \leqslant \frac{1}{\sigma}L = 4L$ for $\widetilde{N}_3 \neq 0$. It follows now from (1) that

$$\widetilde{N}_3 \leqslant 51m^2 8^{4m} 2^{3m+1} A_{d-1}^{2m-2}(16\sqrt{2})^{2m-2} 4L\left(\log_4\left(B_{d-1}\,\frac{1}{2\sqrt{2}}\,8^{2m}L\right)\right)^{(2m-2)r_{d-1}}$$
$$= \frac{51}{2^6}\,m^2\,2^{24m}A_{d-1}^{2m-2}L\left(\log_4\left(B_{d-1}\,\frac{1}{2\sqrt{2}}\,8^{2m}L\right)\right)^{(2m-2)r_{d-1}}$$

for $\widetilde{N}_3 \neq 0$. We denote the right-hand side of the resulting inequality by $D(L)$. Clearly, $D(L) > 0$ for $0 \leqslant k(L,\sigma,C;n_1)$, so that $\widetilde{N}_3 \leqslant D(L)$ for all $k_0,\dots,k_n$ such that $0 \leqslant k_i \leqslant k(L,\sigma,C;n_1)$ (and not only for the values of k_i such that $\widetilde{N}_3 \neq 0$).

We now proceed to the case $n = 0$. Here the estimate (1) involves no factors of the form $\widetilde{N}\left(\frac{\sqrt{2}}{\sigma}\frac{L_i}{L_{i+1}}\right)$ and, moreover, $n_1 = 0$ and $L_0 \leqslant L$. As a result, we obtain

$$\widetilde{N}_3 \leqslant 51m^2 \cdot 8^{4m-n_1} \cdot 2^{3m+1}L_0 \leqslant 2 \cdot 51m^2 \cdot 2^{15m}L.$$

The quantity $D(L)$ is the right-hand side of this inequality multiplied by

$$\frac{2^{9m}}{2^7} \cdot A_{d-1}^{2m-2} \cdot \left(\log_4\left(B_{d-1}\,\frac{1}{2\sqrt{2}}\,8^{2m}L\right)\right)^{(2m-2)r_{d-1}}.$$

In our case $k(L,\sigma,C;n_1)=\log_4(4L)$, therefore

$$\log_4\Big(B_{d-1}\frac{1}{2\sqrt{2}}8^{2m}L\Big)\geqslant\log_4\Big(\frac{1}{2\sqrt{2}}8^2L\Big)>1$$

because $k(L,\sigma,C;n_1)\geqslant 0$. In addition, $2^{9m}>2^6$ and $A_{d-1}^{2m-2}\geqslant 1$, so that if $n=0$, then also $\widetilde{N}_3\leqslant D(L)$ for $0\leqslant k_0\leqslant k(L,\sigma,C;n_1)$.

We now assume that $L\geqslant 1$. Then $k(L,\sigma,C;n_1)>0$ for each $n_1\geqslant 0$. Since the value of $D(L)$ is independent of $k_0,\dots,k_n$ and n,n_1,n_2, we can use Corollary 4.9 and then Lemma 4.7 to obtain

$$\begin{aligned}N_3(K,\Gamma;C,\delta,L;n,n_1,n_2)&\leqslant D(L)\cdot\big(k(L,\sigma,C;n_1)+1\big)^{n+1}\\&=D(L)\cdot\big(\log_4(4\cdot 8^{n_1}L)+1\big)^{n+1}\\&\leqslant D(L)\cdot\Big(\log_4\Big(\frac{1}{4}\cdot 8^{2m}L\Big)\Big)^{2m-1}\end{aligned}$$

and

$$N_2(K,\Gamma;L)\leqslant D(L)\cdot\Big(\log_4\Big(\frac{1}{4}\cdot 8^{2m}L\Big)\Big)^{2m-1}\cdot(2m)^3.$$

Finally, $L+1\leqslant 2L$ for $L\geqslant 1$, therefore

$$\begin{aligned}N_1(K,\Gamma;L)&\leqslant 3\cdot N_2(K,\Gamma;2L)\leqslant 3\cdot D(2L)\cdot\Big(\log_4\Big(\frac{1}{4}8^{2m}\,2L\Big)\Big)^{2m-1}(2m)^3\\&=3\,\frac{51}{2^6}\,m^2\,2^{24m}A_{d-1}^{2m-2}\,2L\,\Big(\log_4\Big(B_{d-1}\frac{1}{2\sqrt{2}}8^{2m}\,2L\Big)\Big)^{(2m-2)r_{d-1}}\\&\quad\times\Big(\log_4\Big(\frac{1}{4}8^{2m}\,2L\Big)\Big)^{2m-1}(2m)^3\\&\leqslant 40m^5\cdot 2^{24m}\cdot A_{d-1}^{2m-2}L\big(\log_4(B_{d-1}\cdot 2^{6m}L)\big)^{(2m-2)r_{d-1}+(2m-1)}\end{aligned}$$

by Lemma 4.6. We note that $B_{d-1}\cdot 2^{6m}=B_d$ and $(2m-2)\,r_{d-1}+(2m-1)<r_d$ for $d-1>1$. Hence to complete the proof of the induction step it suffices to show that $40m^5\cdot 2^{24m}\cdot A_{d-1}^{2m-2}\leqslant A_d$.

The function $f(x)=(40x^5)^{1/4x}$ increases for $0<x\leqslant 40^{-1/5}e<2$ and decreases for $x\geqslant 40^{-1/5}e$. Hence $(40m^5)^{1/4m}\leqslant\max(40^{1/4},(40\cdot 2^5)^{1/8})<3$. Consequently,

$$\begin{aligned}40m^5\cdot 2^{24m}\cdot A_{d-1}^{2m-2}&=40m^5\cdot 2^{24m}\cdot A_{d-1}^{-2}\cdot A_d=\frac{40m^5\cdot 2^{24m}}{(3\cdot 2^6)^{2(2m)^{d-1}}}A_d\\&\leqslant\frac{40m^5\cdot 2^{24m}}{(3\cdot 2^6)^{4m}}A_d=\Big(\frac{1}{3}\,(40m^5)^{1/4m}\Big)^{4m}A_d<A_d\end{aligned}$$

for $d-1\geqslant 1$, as required.

As regards the case of $d-1=1$, using arguments similar to the above we can obtain the following estimate:

$$N_1(K,\Gamma;L)\leqslant 40m^5\cdot 2^{24m}\cdot A_1^{2m-2}L\Big(\log_4\Big(\frac{1}{2}\cdot 2^{6m}L\Big)\Big)^{2m-1}\quad\text{for } L\geqslant 1.$$

Since $\frac{1}{2} \cdot 2^{6m} < B_2$, $2m-1 < r_2$, and $40m^5 \cdot 2^{24m} \cdot A_1^{2m-2} < A_2$ (as shown above), we have proved the induction step also in that case.

Proof of Theorem 4.1. Let ω' be a flat structure that is homothetic to ω with coefficient λ. Then ω and ω' have the same saddle connections and the length of a saddle connection with respect to ω' is λ times its length with respect to ω. Hence we obtain easily that the theorem must either hold or fail for both structures, so that it suffices to consider only the case of $s = 1$.

The proof that follows is very similar to that of Theorem 4.15.

For arbitrary $C, \delta, L > 0$ and integers n, n_1, n_2, where $0 \leqslant n, n_1, n_2 < 2m$, let $N_3'(C, \delta, L; n, n_1, n_2)$ be the number of collections consisting of pairwise disjoint saddle connections $\gamma_0, \dots, \gamma_n$ and a complex $\widetilde{K}$ for which there exist a direction v, integers $d_0, \dots, d_n$, and real numbers $l_0, \dots, l_n$ such that

(a) conditions (1)–(6) in Proposition 4.5 hold with the entire surface M regarded as the ambient complex K;

(b) $d_n = n_1$ and $d(\widetilde{K}) = n_2$;

(c) $|\gamma_0| \leqslant L$.

By Proposition 4.5,

$$N(L) \leqslant \sum_{0 \leqslant n, n_1, n_2 < 2m} N_3'(C, \delta, L; n, n_1, n_2) \quad \text{for each } L > 0, \tag{2}$$

if $\delta \leqslant l^2_{\min}(C)$ and $\delta \leqslant \dfrac{\operatorname{Area}(M)}{2m(C+2)^{4m}}$. Next, given integers $k_0, \dots, k_n \geqslant 0$ and a real number σ, $0 < \sigma < 1$, let $\widetilde{N}_3'(C, \delta, L, \sigma; n, n_1, n_2; k_0, \dots, k_n)$ be the number of collections $\gamma_0, \dots, \gamma_n, \widetilde{K}$ satisfying the additional condition

(d) $\sigma L_i \leqslant v(\gamma_i) \leqslant L_i$, where $L_i = \sigma^{k_i}(C+2)^{2m-1}L$, $0 \leqslant i \leqslant n$.

Assume now that saddle connections $\gamma_0, \dots, \gamma_n$ and a complex $\widetilde{K}$ satisfy (a)–(c). Then $v(\gamma_i) = (C+2)^{d_i} l_i \leqslant (C+2)^{n_1}|\gamma_0| \leqslant (C+2)^{2m-1}L$, that is, (d) holds for some $k_0, \dots, k_n \geqslant 0$. By Lemma 4.4 we have $v(\gamma_i) \geqslant \frac{1}{\sqrt{2}}|\gamma_i| \geqslant \frac{s}{\sqrt{2}} = \frac{1}{\sqrt{2}}$ for $\delta \leqslant l^2_{\min}(C)$, and therefore $L_i \geqslant \frac{1}{\sqrt{2}}$. By (c) and (d),

$$L_0 \leqslant \frac{1}{\sigma}|\gamma_0| \leqslant \frac{1}{\sigma}L.$$

Hence

$$L_n \geqslant \frac{1}{\sqrt{2}} \quad \text{and} \quad L_0 \leqslant \frac{L}{\sigma}. \tag{3}$$

Further,

$$\frac{v(\gamma_i)}{v(\gamma_{i+1})} = \frac{l_i}{l_{i+1}}(C+2)^{d_i - d_{i+1}}$$

for $0 \leqslant i < n$. By the inequalities $1 \leqslant l_i/l_{i+1} \leqslant |\gamma_0|/l_n$ and condition (d) we obtain first

$$\frac{L_{i+1}}{L_i} \leqslant \frac{1}{\sigma}(C+2)^{d_{i+1}-d_i},$$

and therefore $\prod_{i=0}^{n-1}$

Second,

$\frac{L_i}{L_{i+1}} \leqslant \frac{1}{\sigma}(C$

$\leqslant \frac{1}{\sigma}(C$

which, in view of the con

$L_i/L_{i+1} \leqslant \sigma^{-2}(C$

Finally,

$\frac{L_n}{L_i} \leqslant \frac{1}{\sigma}\frac{v}{v}$

for $0 \leqslant i < n$, and since σ

$0 \leqslant k_i \leqslant$

$=$

Hence

$N_3'(C, \delta, L; n, n_1$

for $\delta \leqslant l^2_{\min}(C)$, where t

(3)-(6) hold.

Let $S_4'(C, \delta, \sigma, L; n_1)$ b

exists l, $\sigma L \leqslant l \leqslant L$, su

some direction v. Let N_4'

similar way to Lemma 4.

$\widetilde{N}_3'(C, \delta, L, \sigma; n, n_1, n$

for $\delta \leqslant l^2_{\min}(C)$ and C

$N_1(K, \Gamma; \widetilde{L}) \leqslant \widetilde{N}'(\widetilde{L})$ for

at the boundary such th

$\delta \leqslant l^2_{\min}(C)$ we have m (

by Theorem 4.15.

and therefore

$$\prod_{i=0}^{n-1} \max\left(1, \frac{L_{i+1}}{L_i}\right) \leqslant \left(\frac{1}{\sigma}\right)^n (C+2)^{n_1}. \tag{4}$$

Second,

$$\begin{aligned}\frac{L_i}{L_{i+1}} &\leqslant \frac{1}{\sigma}(C+2)^{d_i-d_{i+1}}\frac{|\gamma_0|}{l_n} \leqslant \frac{1}{\sigma}(C+2)^{d_i-d_{i+1}+n_1}\frac{|\gamma_0|}{v(\gamma_n)} \\ &\leqslant \frac{1}{\sigma}(C+2)^{n_1-1}\frac{L}{v(\gamma_n)} \leqslant \frac{1}{\sigma}(C+2)^{2m-2}\frac{L}{v(\gamma_n)},\end{aligned}$$

which, in view of the condition $v(\gamma_n) \geqslant \sigma L_n$, shows that

$$L_i/L_{i+1} \leqslant \sigma^{-2}(C+2)^{-1}\cdot(C+2)^{2m-1}L/L_n = \sigma^{-2}(C+2)^{-1}\,\sigma^{-k_n}. \tag{5}$$

Finally,

$$\frac{L_n}{L_i} \leqslant \frac{1}{\sigma}\,\frac{v(\gamma_n)}{v(\gamma_i)} = \frac{1}{\sigma}(C+2)^{d_n-d_i}\frac{l_n}{l_i} \leqslant \frac{1}{\sigma}(C+2)^{2m-1}$$

for $0 \leqslant i < n$, and since $\sigma^{-k_i} = (C+2)^{2m-1}L/L_i$, it follows that

$$\begin{aligned}0 \leqslant k_i &\leqslant \log_{1/\sigma}\left(\frac{1}{\sigma}(C+2)^{2m-1}\cdot\frac{(C+2)^{2m-1}L}{L_n}\right) \\ &= \log_{1/\sigma}\left(\sigma^{-1}(C+2)^{2m-1}\sigma^{-k_n}\right).\end{aligned} \tag{6}$$

Hence

$$N_3'(C,\delta,L;n,n_1,n_2) \leqslant \sum_{k_0,\dots,k_n} \widetilde{N}_3'(C,\delta,L,\sigma;n,n_1,n_2;k_0,\dots,k_n) \tag{7}$$

for $\delta \leqslant l_{\min}^2(C)$, where the sum is taken over those values of $k_0,\dots,k_n$ such that (3)–(6) hold.

Let $S_4'(C,\delta,\sigma,L;n_1)$ be the set of saddle connections for each of which there exists l, $\sigma L \leqslant l \leqslant L$, such that the connection is (l,δ,C,n_1)-insulated relative to some direction v. Let $N_4'(C,\delta,\sigma,L;n_1)$ be the number of elements in this set. In a similar way to Lemma 4.14 we can prove that

$$\begin{aligned}&\widetilde{N}_3'(C,\delta,L,\sigma;n,n_1,n_2;k_0,\dots,k_n) \\ &\qquad \leqslant N_4'(C,\delta,\sigma,L_n;n_1)\cdot 2^{3m+1}\cdot\prod_{i=0}^{n-1}\widetilde{N}'\left(\frac{\sqrt{2}}{\sigma}\,\frac{L_i}{L_{i+1}}\right)\end{aligned} \tag{8}$$

for $\delta \leqslant l_{\min}^2(C)$ and $C \geqslant 4\sqrt{2}$, where $\widetilde{N}'$ is a function satisfying the inequality $N_1(K,\Gamma;\widetilde{L}) \leqslant \widetilde{N}'(\widetilde{L})$ for $\widetilde{L} > 0$ and for each complex K with a saddle connection Γ at the boundary such that $0 < d(K) < 2m$ and $\mathrm{Area}(K) \leqslant m\,(C+2)^{4m}\delta$. For $\delta \leqslant l_{\min}^2(C)$ we have $m\,(C+2)^{4m}\delta \leqslant m{\cdot}s^2/2 < m{\cdot}s^2$, therefore we can set

$$\widetilde{N}'(\widetilde{L}) = \widetilde{N}_1\left(2m-1;\max(1,\widetilde{L})\right) \tag{9}$$

by Theorem 4.15.

If γ and $\widetilde{\gamma}$ are intersecting saddle connections in $S_4'(C,\delta,\sigma,L;n_1)$, then

$$\angle(\gamma,\widetilde{\gamma}) > \delta\left(\frac{C}{\sqrt{2}}-1\right)\frac{(C+2)^{2n_1}}{L^2}$$

for $\delta \leqslant l_{\min}^2(C)$ and $C\sigma \geqslant \sqrt{2}$. This inequality can be proved in the same way as the one in Lemma 4.10. Let φ_0 be the right-hand side of this inequality. Saddle connections in $S_4'(C,\delta,\sigma,L;n_1)$ with directions belonging to a fixed arc of length φ_0 are disjoint, therefore there are at most $3m$ of them by Proposition 4.2. Averaging over all the arcs of length φ_0 and bearing in mind that there are two opposite directions corresponding to each saddle connection we see that

$$\begin{aligned} N_4'(C,\delta,\sigma,L;n_1) &\leqslant 3m\cdot\max\left(\frac{\pi}{\varphi_0},1\right)\\ &= 3m\cdot\max\left(1,\frac{\pi}{\delta(\frac{C}{\sqrt{2}}-1)(C+2)^{2n_1}}L^2\right) \qquad (10)\end{aligned}$$

for $\delta \leqslant l_{\min}^2(C)$ and $C\sigma \geqslant \sqrt{2}$.

We now fix constants C,σ, and δ. We set $C=6$, $\sigma=1/4$, and $\delta=\dfrac{1}{4m\cdot 8^{4m}}$. Then $C\sigma\geqslant\sqrt{2}$, $C\geqslant 4\sqrt{2}$, moreover, $\delta\leqslant l_{\min}^2(C)=\dfrac{s^2}{2(C+2)^{4m}}$ since $s=1$, and $\delta\leqslant\dfrac{\mathrm{Area}(M)}{2m(C+2)^{4m}}$ since $\mathrm{Area}(M)\geqslant s^2/2=1/2$ by Lemma 3.7. Hence we can apply inequalities (2), (7), (8), and (10). In particular, it follows from (10) that

$$\begin{aligned} N_4'(C,\delta,\sigma,L;n_1) &\leqslant 3m\cdot\max\left(1,\frac{\pi}{6/\sqrt{2}-1}\cdot 4m\cdot 8^{4m-2n_1}L^2\right)\\ &\leqslant 3m\cdot\max\left(1,4m\cdot 2^{12m-6n_1}L^2\right)\\ &= 12m^2\cdot 2^{12m-6n_1}L^2 \quad\text{for}\quad L\geqslant\frac{1}{\sqrt{2}}.\end{aligned}$$

We now find an estimate for $\widetilde{N}_3'=\widetilde{N}_3'(C,\delta,L,\sigma;n,n_1,n_2;k_0,\dots,k_n)$ under the assumption that (3)–(6) hold. We substitute the above estimate for N_4' in (8) (this is possible in view of (3)). We obtain

$$\widetilde{N}_3' \leqslant 24m^2\cdot 2^{15m-6n_1}L_n^2\cdot\prod_{i=0}^{n-1}\widetilde{N}'\left(4\sqrt{2}\,\frac{L_i}{L_{i+1}}\right). \qquad (11)$$

Let $n>0$ and $2m-1>1$. By (9),

$$\widetilde{N}'\left(4\sqrt{2}\,\frac{L_i}{L_{i+1}}\right)\leqslant A_{2m-1}\cdot 4\sqrt{2}\,\frac{L_i}{L_{i+1}}\left(\log_4\left(B_{2m-1}\cdot 4\sqrt{2}\,\frac{L_i}{L_{i+1}}\right)\right)^{r_{2m-1}}$$

if $\dfrac{L_i}{L_{i+1}}\geqslant 1$ and

$$\widetilde{N}'\left(4\sqrt{2}\,\frac{L_i}{L_{i+1}}\right)\leqslant A_{2m-1}\cdot 4\sqrt{2}\left(\log_4(B_{2m-1}\cdot 4\sqrt{2})\right)^{r_{2m-1}}$$

if $\dfrac{L_i}{L_{i+1}} < 1$.

In view of (5),

$$L_i/L_{i+1} \leqslant \frac{1}{\sigma^2(C+2)}\,\sigma^{-k_n} = 2\cdot 4^{k_n}.$$

We even have $L_i/L_{i+1} \leqslant 4^{k_n}$ since L_i/L_{i+1} is an integer power of $1/\sigma = 4$. Together with the inequality $1 \leqslant 4^{k_n}$ this shows that

$$\begin{aligned}
&\prod_{i=0}^{n-1} \widetilde{N}'\left(4\sqrt{2}\,\frac{L_i}{L_{i+1}}\right)\\
&\qquad \leqslant A_{2m-1}^{n}\,(4\sqrt{2})^n\,\frac{L_0}{L_n}\prod_{i=0}^{n-1}\max\left(1,\frac{L_{i+1}}{L_i}\right)\left(\log_4(B_{2m-1}\cdot 4\sqrt{2}\,4^{k_n})\right)^{nr_{2m-1}}.
\end{aligned}$$

Bearing in mind (3) and (4) we obtain

$$\begin{aligned}
&\prod_{i=0}^{n-1} \widetilde{N}'\left(4\sqrt{2}\,\frac{L_i}{L_{i+1}}\right)\\
&\qquad \leqslant A_{2m-1}^{2m-1}\,(4\sqrt{2})^n\,4\frac{L}{L_n}(4^n\,8^{n_1})\left(k_n + 3m(2m-2) + \frac{5}{4}\right)^{(2m-1)r_{2m-1}}.
\end{aligned}$$

Substituting this in (11) we see that

$$\begin{aligned}
\widetilde{N}_3' &\leqslant 24m^2\,2^{15m-6n_1}L_n^2\,A_{2m-1}^{2m-1}\,(16\sqrt{2}\,)^n\\
&\qquad \times 4\frac{L}{L_n}\,8^{n_1}\left(k_n + 3m(2m-2) + \frac{5}{4}\right)^{(2m-1)r_{2m-1}}\\
&\leqslant 96m^2\,2^{15m}\,(2\sqrt{2}\,)^n A_{2m-1}^{2m-1}\frac{L_n}{L}\,L^2(k_n + 6m^2)^{(2m-1)r_{2m-1}}.
\end{aligned}$$

Finally, $L_n/L = (C+2)^{2m-1}\sigma^{k_n} = 8^{2m-1}\cdot 4^{-k_n}$ and therefore

$$\widetilde{N}_3' \leqslant \frac{96}{2\sqrt{2}\cdot 8}\,m^2\,2^{24m}A_{2m-1}^{2m-1}L^2\cdot 4^{k_n}(k_n+6m^2)^{(2m-1)r_{2m-1}}.$$

By (6) we obtain

$$0 \leqslant k_i \leqslant \log_{1/\sigma}\left(\sigma^{-1}(C+2)^{2m-1}\,\sigma^{-k_n}\right) = \log_4\left(\frac{1}{2}\,8^{2m}\cdot 4^{k_n}\right) = k_n + 3m - \frac{1}{2}$$

for $0 \leqslant i < n$, that is, the k_i can take at most $k_n + 3m$ different values. Our estimate of $\widetilde{N}_3'$ is independent of $k_0,\dots,k_{n-1}$, therefore

$$\begin{aligned}
N_3' &= N_3'(C,\delta,L;n,n_1,n_2)\\
&\leqslant \frac{96}{2\sqrt{2}\cdot 8}\,m^2\,2^{24m}A_{2m-1}^{2m-1}\,L^2\cdot\sum_{k_n=0}^{\infty}\left(4^{-k_n}\,(k_n+6m^2)^{(2m-1)\,r_{2m-1}}(k_n+3m)^n\right)\\
&\leqslant \frac{96}{2\sqrt{2}\cdot 8}\,m^2\,2^{24m}A_{2m-1}^{2m-1}\,L^2\cdot\sum_{k=0}^{\infty}\left(4^{-k}(k+6m^2)^{(2m-1)r_{2m-1}+(2m-1)}\right)
\end{aligned}$$

by (7). For $2m-1>1$ we have $(2m-1)\,r_{2m-1}+(2m-1)<r_{2m}=(2m)^{2m-1}$. Hence

$$N_3' \leqslant \frac{96}{2\sqrt{2}\cdot 8}\, m^2\, 2^{24m} A_{2m-1}^{2m-1}\, L^2 \cdot \sum_{k=0}^{\infty}\Bigl(4^{-k}(k+6m^2)^{(2m)^{2m-1}}\Bigr). \tag{12}$$

We now consider the case of $n>0$ and $2m-1=1$. In this case there is no logarithmic factor in the expression for $\widetilde{N}'$, but our arguments can be the same as above in all other respects. Our final result is as follows:

$$N_3' \leqslant \frac{96}{2\sqrt{2}\cdot 8}\, m^2\, 2^{24m} A_{2m-1}^{2m-1}\, L^2 \cdot \sum_{k=0}^{\infty}\bigl(4^{-k}(k+3m)^{2m-1}\bigr).$$

The estimate (12) is an obvious consequence of this inequality.

As for $n=0$, there are no factors of the form $\widetilde{N}'\Bigl(4\sqrt{2}\,\dfrac{L_i}{L_{i+1}}\Bigr)$ in (11) in that case and we immediately obtain

$$\widetilde{N}_3' \leqslant 24m^2 \cdot 2^{15m-6n_1} \cdot L_0^2 = 24m^2 \cdot 2^{15m}(L_0/L)^2 L^2.$$

However, $L_0/L=(C+2)^{2m-1}\sigma^{k_0}=\frac{1}{8}\,2^{6m}\cdot 4^{-k_0}$, so that $\widetilde{N}_3' \leqslant \frac{24}{8^2}\, m^2\cdot 2^{27m}\, L^2\cdot 4^{-2k_0}$ and

$$N_3' \leqslant \frac{24}{8^2}\, m^2 \cdot 2^{27m} L^2 \cdot \sum_{k_0=0}^{\infty} 4^{-2k_0}$$

by (7). Since

$$\frac{24}{8^2}\, 2^{3m} < \frac{96}{2\sqrt{2}\cdot 8}\, A_{2m-1}^{2m-1},$$

while $4^{-2k_0}\leqslant 4^{-k_0}$ and $k_0+6m^2>1$ for $k_0\geqslant 0$, the estimate (12) remains valid in this case.

Bearing in mind that (12) is independent of the triple of parameters n, n_1, n_2, by (2) we obtain

$$N(L) \leqslant \frac{96}{2\sqrt{2}}\, m^5\, 2^{24m} A_{2m-1}^{2m-1}\, L^2 \cdot \sum_{k=0}^{\infty}\Bigl(4^{-k}(k+6m^2)^{(2m)^{2m-1}}\Bigr). \tag{13}$$

Next we consider two cases: $m>1$ and $m=1$. Assume that $m>1$. In our proof of Theorem 4.15 we established that $40m^5\leqslant 3^{4m}$. Since $\frac{96}{2\sqrt{2}}<40$, it follows that $\frac{96}{2\sqrt{2}}\, m^5\, 2^{24m}\leqslant (3\cdot 2^6)^{4m}$. Further,

$$(3\cdot 2^6)^{4m}\cdot A_{2m-1}^{2m-1} = (3\cdot 2^6)^{4m+(2m)^{2m-1}(2m-1)} < (3\cdot 2^6)^{(2m)^{2m}} \quad \text{for } m>1,$$

so that

$$N(L) \leqslant (3\cdot 2^6)^{(2m)^{2m}} \cdot L^2 \cdot \sum_{k=0}^{\infty}\Bigl(4^{-k}(k+6m^2)^{(2m)^{2m-1}}\Bigr).$$

We now find an estimate of the sum of this series. For $0 \leqslant k \leqslant 6m^2$ we have $k + 6m^2 \leqslant 12m^2$, while $k + 6m^2 < 2k$ for $k > 6m^2$; hence

$$\begin{aligned}\sum_{k=0}^{\infty}\Big(4^{-k}\,(k+6m^2)^{(2m)^{2m-1}}\Big) &\leqslant \sum_{k=0}^{\infty}4^{-k}\Big((12m^2)^{(2m)^{2m-1}} + (2k)^{(2m)^{2m-1}}\Big)\\ &\leqslant \frac{4}{3}(12m^2)^{(2m)^{2m-1}} + 2^{(2m)^{2m-1}}\cdot\sum_{k=0}^{\infty}\Big(e^{-k}\cdot k^{(2m)^{2m-1}}\Big).\end{aligned}$$

We denote $(2m)^{2m-1}$ by p. The function $g(x) = e^{-x}x^p$ increases for $0 \leqslant x \leqslant p$ and decreases for $x \geqslant p$. Hence

$$\sum_{k=0}^{\infty} e^{-k}\,k^p \leqslant \int_0^{\infty} e^{-x}x^p\,dx + e^{-p}p^p = p! + e^{-p}p^p,$$

which is less than p^p for $p > 1$. Thus,

$$\begin{aligned}\sum_{k=0}^{\infty}\Big(4^{-k}\,(k+6m^2)^{(2m)^{2m-1}}\Big) &\leqslant \frac{4}{3}(12m^2)^{(2m)^{2m-1}} + 2^{(2m)^{2m-1}}\cdot\Big((2m)^{2m-1}\Big)^{(2m)^{2m-1}}\\ &= (2m)^{(2m)^{2m}}\left(\frac{4}{3}\left(\frac{3}{(2m)^{2m-2}}\right)^{(2m)^{2m-1}} + \left(\frac{1}{m}\right)^{(2m)^{2m-1}}\right)\\ &< (2m)^{(2m)^{2m}} \quad \text{for} \quad m > 1.\end{aligned}$$

As a result,

$$N(L) \leqslant (3\cdot 2^6)^{(2m)^{2m}}\cdot(2m)^{(2m)^{2m}}\cdot L^2 < (400m)^{(2m)^{2m}}\cdot L^2,$$

as required.

For $m = 1$ we obtain

$$N(L) \leqslant \frac{96}{2\sqrt{2}}\cdot 2^{24}\cdot(3\cdot 2^6)^2\cdot L^2\cdot\sum_{k=0}^{\infty}4^{-k}(k+6)^2$$

by inequality (13). The sums $\sum_{k=0}^{\infty}4^{-k}$, $\sum_{k=0}^{\infty}(4^{-k}\cdot k)$, and $\sum_{k=0}^{\infty}(4^{-k}\cdot k^2)$ are equal to $\Sigma_0 = 4/3$, $\Sigma_1 = 4/9$, and $\Sigma_2 = 20/27$, respectively, therefore the sum $\sum_{k=0}^{\infty}4^{-k}(k+6)^2$ is equal to $\Sigma_2 + 12\Sigma_1 + 36\Sigma_0 = 52\frac{2}{27}$. Hence

$$N(L) \leqslant \frac{96}{2\sqrt{2}}\cdot 2^{24}\cdot(3\cdot 2^6)^2\cdot 52\frac{2}{27}\cdot L^2 < (3\cdot 2^7)^6\cdot L^2,$$

which completes the proof in this case.

Bibliography

[1] A. N. Zemlyakov [Zemljakov] and A. B. Katok, "Topological transitivity of billiards in polygons", *Mat. Zametki* **18** (1975), 291–300; English transl. in *Math. Notes* **18** (1975).

[2] S. Kerckhoff, H. Masur, and J. Smillie, "Ergodicity of billiard flows and quadratic differentials", *Ann. of Math.* **124** (1986), 293–311.

[3] F. Pospiech and A. M. Stepin, "Über die vollständige Nichtintegrierbarkeit eines Systems zweier Teilchen auf einem Intervall", *Wiss. Z. Univ. Halle* **37**:4 (1988), 89–98.

[4] H. Masur, "The growth rate of trajectories of a quadratic differential", *Ergodic Theory Dynamical Systems* **10**:1 (1990), 151–176.

[5] M. Boshernitzan, "A condition for minimal interval exchange maps to be uniquely ergodic", *Duke Math. J.* **52** (1985), 723–752.

[6] Ya. B. Vorobets, "Billiard ergodicity in polygons: explicit examples", *Uspekhi Mat. Nauk* **51**:4 (1996), 151–152; English transl. in *Russian Math. Surveys* **51**:4 (1996).

Moscow State University

Received 20/JUN/96
Translated by IPS (DoM)

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Sbornik: Mathematics **188**:3 43
Matematicheskiĭ Sbornik **188**:3

Absorbin
in absolu

Abstract. Absorbing set
for n-dimensional separab
Bibliography: 25 titles.

The method of absorbi
topological classification, c
manifolds modelled in pre-
Bessaga and Pelczynski [2
Bestvina and Mogilski [4].
of manifolds modelled on t
the absolute Borel spaces o
has obtained analogous re

Recently, the infinite-d
the finite-dimensional cas
We recall that the first
dimensional manifolds was
(Bestvina [7], Dranishniko
spaces the Menger cube μ
of compact metrizable spa

The model spaces Λ_α a
For example, Λ_1 is homeo

rint Q =

of Q. The realizations in
There arises the natural
and Ω_α. A more precise s
property of Λ_α and Ω_α th
Borel classes. This can be

Until recently, results
σ-compact spaces (that is
see [10]–[12]. This class li
absorbing sets constructe
the Hilbert space, or in th

AMS 1991 *Mathematics Su*