# On the topological full group containing the Grigorchuk group

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**Abstract**

We consider the topological full group of a substitution subshift induced by a substitution $a \to aca$, $b \to d$, $c \to b$, $d \to c$. This group is interesting since the Grigorchuk group naturally embeds into it. We show that the topological full group is finitely generated and give a simple generating set for it.

## 1 Introduction

Let $X$ be a Cantor set and $T : X \to X$ be a minimal homeomorphism. The *topological full group* of $T$, denoted $[[T]]$, is a transformation group consisting of all homeomorphisms $f : X \to X$ that can be given by $f(x) = T^{\nu(x)}(x)$, $x \in X$ for some continuous function $\nu : X \to \mathbb{Z}$. Continuity of $\nu$ implies that this function is locally constant and takes only finitely many values. Then nonempty level sets of $\nu$ form a partition of the Cantor set $X$ into clopen (i.e., both closed and open) sets. Thus every element of $[[T]]$ is "piecewise" a power of $T$. The topological full group $[[T]]$ is countable (as there are only countably many clopen subsets of $X$).

The notion of the topological full group was introduced by Giordano, Putnam and Skau [GPS] who showed that $[[T]]$ is an (almost) complete invariant of $T$ as a topological dynamical system.

**Theorem 1.1 ([GPS])** *Given minimal homeomorphisms $T_1 : X \to X$ and $T_2 : X \to X$ of a Cantor set $X$, the topological full groups $[[T_1]]$ and $[[T_2]]$ are isomorphic if and only if $T_1$ is topologically conjugate to $T_2$ or $T_2^{-1}$.*

In this paper we are concerned with group-theoretical properties of a topological full group $[[T]]$.

**Theorem 1.2 ([GPS])** *There exists a unique homomorphism $I : [[T]] \to \mathbb{Z}$ such that $I(T) = 1$.*

The homomorphism $I$ is called the *index map*. Clearly, every element of finite order is contained in the kernel of $I$.

**Theorem 1.3 ([Mat])** *The kernel of the index map is generated by elements of finite order.*

Theorem 1.3 was proved by Matui [Mat] who initiated the systematic study of group-theoretic properties of topological full groups (he also introduced the notation $[[T]]$).

We can construct many elements of finite order in $[[T]]$ as follows. Let $U \subset X$ be a clopen set. Suppose that for some integers $M$ and $N$, $M < N$, the sets $T^M(U), T^{M+1}(U), \ldots, T^N(U)$ are pairwise disjoint. Then one can define a transformation $\Psi_{U,M,N} : X \to X$ by

$$\Psi_{U,M,N}(x) = \begin{cases} T(x) & \text{if } x \in T^M(U) \cup T^{M+1}(U) \cup \ldots \cup T^{N-1}(U), \\ T^{M-N}(x) & \text{if } x \in T^N(U), \\ x & \text{otherwise.} \end{cases}$$

By construction, $\Psi_{U,M,N}$ is an element of the topological full group $[[T]]$ of order $N - M + 1$. We are also going to use alternative notation $\delta_U$ for the map $\Psi_{U,0,1}$ and $\tau_U$ for the map $\Psi_{U,0,2}$. Each $\delta_U$ is an element of order 2 while each $\tau_U$ is an element of order 3 (hence the notation: $\delta$ as in $\delta\acute{\upsilon}o$, $\tau$ as in $\tau\rho\acute{\iota}\alpha$).

It is not hard to show that every element of finite order in $[[T]]$ can be decomposed as a product of elements of the form $\delta_U$. Together with Theorems 1.2 and 1.3, this yields the following.

**Theorem 1.4 ([Mat])** *The topological full group $[[T]]$ is generated by $T$ and all transformations of the form $\delta_U$.*

There is much more to say about the commutator group of $[[T]]$.

**Theorem 1.5 ([Mat])** *The commutator group of $[[T]]$ is generated by all elements of the form $\tau_U$.*

**Theorem 1.6 ([Mat])** *The commutator group of $[[T]]$ is simple.*

**Theorem 1.7 ([Mat])** *The commutator group of $[[T]]$ is finitely generated if and only if $T$ is topologically conjugate to a (minimal) subshift.*

Now we introduce a specific transformation $T$, a substitution subshift. Let $\sigma$ denote the *Lysenok substitution* over the alphabet $\mathcal{A} = \{a, b, c, d\}$, namely, $\sigma(a) = aca$, $\sigma(b) = d$, $\sigma(c) = b$, and $\sigma(d) = c$. This substitution was originally used by Lysenok [Lys] to obtain a nice recursive presentation of the Grigorchuk group:

$$\mathcal{G} = \langle a, b, c, d \mid 1 = a^2 = b^2 = c^2 = d^2 = bcd = \sigma^k((ad)^4) = \sigma^k((adacac)^4),\ k \geq 0 \rangle.$$

The substitution $\sigma$ acts naturally on the set $\mathcal{A}^*$ of finite words over the alphabet $\mathcal{A}$ as well as on the set $\mathcal{A}^{\mathbb{N}}$ of infinite words over $\mathcal{A}$. There exists a unique infinite word $\xi \in \mathcal{A}^{\mathbb{N}}$ fixed by $\sigma$: $\xi = acabacad\ldots$ Let $T : \Omega \to \Omega$ be the two-sided subshift generated by $\xi$. The phase space $\Omega$ of the subshift $T$ consists of bi-infinite sequences $\omega = \ldots\omega_{-2}\omega_{-1}\omega_0.\omega_1\omega_2\ldots$ such that every finite subword $\omega_l\omega_{l+1}\ldots\omega_{m-1}\omega_m$ occurs somewhere in $\xi$. The transformation is defined by $T(\omega) = \ldots\omega_{-1}\omega_0\omega_1.\omega_2\omega_3\ldots$

**Theorem 1.8 ([Vor])** *The subshift $T$ is a minimal homeomorphism of the Cantor set $\Omega$.*

Given two finite words $u$ and $w$ over the alphabet $\mathcal{A}$, we denote by $[u.w]$ the set of all bi-infinite sequences $\omega = \dots\omega_{-2}\omega_{-1}\omega_0.\omega_1\omega_2\dots$ in $\Omega$ such that $\omega_{-M+1}\dots\omega_{-1}\omega_0 = u$ and $\omega_1\omega_2\dots\omega_N = w$, where $M$ is the length of $u$ and $N$ is the length of $w$ ($M, N \geq 0$). We refer to $[u.w]$ as a *cylinder of dimension* $M + N$. The cylinder is a clopen set. Any clopen subset of $\Omega$ splits into a disjoint union of cylinders of dimension $N$ provided that $N$ is large enough.

The cylinder $[u.w]$ is a nonempty set if and only if the concatenated word $uw$ occurs in $\xi$ infinitely many times. Infinitely many occurrences are required since $\xi$ is an infinite sequence while elements of $\Omega$ are bi-infinite sequences. Actually, $\xi$ is a Toeplitz sequence (see [Vor] or Lemma 2.2 below), which implies that every word occurring in $\xi$ does this infinitely often. If at least one of the words $u$ and $w$ is not empty, then the cylinder $[u.w]$ is disjoint from its image $T([u.w])$ (because there are no double letters in $\xi$) so that the transformation $\delta_{[u.w]}$ is well defined.

A direct relation between the Grigorchuk group $\mathcal{G}$ and the topological full group of the substitution subshift $T$ was established by Matte Bon [M-B] who showed that $[[T]]$ contains a copy of $\mathcal{G}$.

**Theorem 1.9 ([M-B])** *The subgroup of* $[[T]]$ *generated by* $\delta_{[.a]}$, $\delta_{[.b]}\delta_{[.c]}$, $\delta_{[.c]}\delta_{[.d]}$, *and* $\delta_{[.d]}\delta_{[.b]}$ *is isomorphic to the Grigorchuk group.*

Theorem 1.7 implies that the commutator group of $[[T]]$ is finitely generated. The main result of this paper is that the entire group $[[T]]$ is finitely generated. Moreover, we provide an explicit generating set.

**Theorem 1.10** *The topological full group of the substitution subshift* $T$ *is generated by transformations* $T$, $\delta_{[.b]}$, $\delta_{[.d]}$, *and* $\delta_{[.acacac]}$.

Note that $\delta_{[.a]}$ and $\delta_{[.c]}$ are not on the list of generators. It turns out that

$$\begin{aligned}\delta_{[.a]} &= T^{-1}\delta_{[.b]}\delta_{[.d]}T^{-2}\delta_{[.b]}\delta_{[.d]}T^{3}\delta_{[.acacac]}T^{2}\delta_{[.acacac]}T^{-2},\\ \delta_{[.c]} &= T^{-2}\delta_{[.b]}\delta_{[.d]}T^{3}\delta_{[.acacac]}T^{2}\delta_{[.acacac]}T^{-3}\end{aligned}$$

(see Section 4 for details).

The paper is organized as follows. In Section 2 we obtain very detailed information on clopen subsets of the Cantor set $\Omega$. In Section 3 we derive some general properties of topological full groups (slightly generalizing [Mat]). In Section 4 the results of Sections 2 and 3 are applied to prove Theorem 1.10. The proof is loosely modeled upon the proof of Theorem 1.7 in [Mat].

## 2 Combinatorics of the substitution subshift

First we are going to establish some properties of the infinite word $\xi = \xi_1\xi_2\xi_3\dots$ fixed by the Lysenok substitution $\sigma$.

For any integer $n \geq 1$ let $w_n = \sigma^{n-1}(a)$. For example, $w_1 = a$, $w_2 = aca$, $w_3 = acabaca$, $w_4 = acabacadacabaca$, $w_5 = acabacadacabacacacabacadacabaca$. Since the word $w_1 = a$ is a proper beginning of the word $w_2 = aca$, it follows by induction that each $w_n$ is a proper beginning of $w_{n+1}$. Consequently, there exists a unique infinite word $\xi \in \mathcal{A}^{\mathbb{N}}$ such that each $w_n$ is a beginning of $\xi$. It is easy to see that $\xi$ is the only infinite word fixed by $\sigma$.

For any integer $n \geq 1$ let $l_n = \sigma^{n-1}(c)$. Then $l_n = c$ if $n$ leaves remainder 1 under division by 3, $l_n = b$ if $n$ leaves remainder 2 under division by 3, and $l_n = d$ if $n$ is divisible by 3.

**Lemma 2.1** *The word $w_n$ has length $2^n - 1$ and $w_{n+1} = w_n l_n w_n$ for all $n \geq 1$.*

**Proof.** For any $n \geq 1$ we obtain that $w_{n+1} = \sigma^n(a) = \sigma^{n-1}(\sigma(a)) = \sigma^{n-1}(aca) = \sigma^{n-1}(a)\sigma^{n-1}(c)\sigma^{n-1}(a) = w_n l_n w_n$. Since the word $w_1 = a$ has length $1 = 2^1 - 1$, $l_n$ is always a single letter, and $(2^n - 1) + 1 + (2^n - 1) = 2^{n+1} - 1$, it follows by induction that the length of $w_n$ is $2^n - 1$ for all $n \geq 1$. ∎

**Lemma 2.2** *Given an integer $N \geq 1$, let $N = 2^n K$, where $n \geq 0$ and $K$ is odd. Then $\xi_N = a$ if $n = 0$ and $\xi_N = l_n$ if $n \geq 1$.*

**Proof.** Let $S$ be the set of all words of the form $ar_1ar_2 \dots ar_M$, where each $r_i \in \{b, c, d\}$. Since $\sigma(ab) = acad$, $\sigma(ac) = acab$, and $\sigma(ad) = acac$, it follows that the set $S$ is invariant under the action of the substitution $\sigma$. Clearly, $ac \in S$. Then $w_m l_m = \sigma^{m-1}(a)\sigma^{m-1}(c) = \sigma^{m-1}(ac)$ is in $S$ for all $m \geq 1$. Since any beginning of the infinite word $\xi$ is also a beginning of some $w_m$, we obtain that $\xi = as_1as_2\dots$, where each $s_i \in \{b, c, d\}$. In particular, $\xi_N = a$ if and only if $N$ is odd.

Since the infinite word $\xi$ is fixed by the substitution $\sigma$, it follows that for any given $n \geq 1$,

$$\xi = \sigma^{n+1}(\xi) = \sigma^{n+1}(a)\sigma^{n+1}(s_1)\sigma^{n+1}(a)\sigma^{n+1}(s_2)\dots = w_{n+1}s'_1 w_{n+1}s'_2 \dots,$$

where $s'_i = \sigma^{n+1}(s_i)$, $i = 1, 2, \dots$ Note that each $s'_i$ is a single letter from $\{b, c, d\}$. By Lemma 2.1, $w_{n+1} = w_n l_n w_n$ and the length of $w_n$ is $2^n - 1$. Since $\xi = w_n l_n w_n s'_1 w_n l_n w_n s'_2 \dots$, we obtain that $\xi_N = l_n$ for $N = 2^n, 3 \cdot 2^n, 5 \cdot 2^n, \dots$ That is, $\xi_N = l_n$ if $N = 2^n K$, where $K$ is odd. ∎

**Lemma 2.3** *$\sigma(\xi_{2N+1}\xi_{2N+2}\dots\xi_{2N+2M}) = \xi_{4N+1}\xi_{4N+2}\dots\xi_{4N+4M}$ for all $N \geq 0$ and $M \geq 1$. Moreover, if $\sigma(w) = \xi_{4N+1}\xi_{4N+2}\dots\xi_{4N+4M}$ for some $w$, then $w = \xi_{2N+1}\xi_{2N+2}\dots\xi_{2N+2M}$.*

**Proof.** For any $M \geq 1$ the word $\xi_1\xi_2\dots\xi_{2M}$ is a beginning of the infinite word $\xi$. Since $\xi$ is invariant under the substitution $\sigma$, the word $\sigma(\xi_1\xi_2\dots\xi_{2M})$ is another beginning of $\xi$. According to Lemma 2.2, $\xi_i = a$ if and only if $i$ is odd. Hence the word $\xi_1\xi_2\dots\xi_{2M}$ contains $M$ letters $a$ and $M$ other letters. Since $\sigma(a) = aca$ is a word of length 3 while $\sigma(b)$, $\sigma(c)$, and $\sigma(d)$ are single letters, the length of $\sigma(\xi_1\xi_2\dots\xi_{2M})$ is $3M + M = 4M$. We conclude that $\sigma(\xi_1\xi_2\dots\xi_{2M}) = \xi_1\xi_2\dots\xi_{4M}$. This proves the first statement of the lemma in the case $N = 0$. In the case $N \geq 1$, it follows from the above that $\sigma(\xi_1\xi_2\dots\xi_{2N}) = \xi_1\xi_2\dots\xi_{4N}$ and $\sigma(\xi_1\xi_2\dots\xi_{2N+2M}) = \xi_1\xi_2\dots\xi_{4N+4M}$. Since

$$\sigma(\xi_1\xi_2\dots\xi_{2N+2M}) = \sigma(\xi_1\xi_2\dots\xi_{2N})\,\sigma(\xi_{2N+1}\xi_{2N+2}\dots\xi_{2N+2M}),$$

we obtain that $\sigma(\xi_{2N+1}\xi_{2N+2}\dots\xi_{2N+2M}) = \xi_{4N+1}\xi_{4N+2}\dots\xi_{4N+4M}$.

To prove the second statement of the lemma, it is enough to show that the action of the substitution $\sigma$ on finite words is one-to-one, i.e., $\sigma(u_1) \neq \sigma(u_2)$ if $u_1 \neq u_2$. The reason is that neither of the words $\sigma(a), \sigma(b), \sigma(c), \sigma(d)$ is a beginning of another (in particular, neither is empty). Let $u$ be the longest common beginning of the words $u_1$ and $u_2$. If $u = u_1$ then $u_2 = u_1 s u_2'$ for some letter $s$ and word $u_2'$. Since $\sigma(s)$ is not empty, we obtain $\sigma(u_2) = \sigma(u_1)\sigma(s)\sigma(u_2') \neq \sigma(u_1)$. The case $u = u_2$ is treated similarly. Otherwise $u_1 = u s_1 u_1'$ and $u_2 = u s_2 u_2'$, where $s_1, s_2$ are distinct letters and $u_1', u_2'$ are some words. It is no loss to assume that the word $\sigma(s_1)$ is not longer than $\sigma(s_2)$. Since $\sigma(s_1)$ is not a beginning of $\sigma(s_2)$, it follows that $\sigma(u)\sigma(s_1)$ is not a beginning of $\sigma(u)\sigma(s_2)\sigma(u_2') = \sigma(u_2)$. Then $\sigma(u_1) = \sigma(u)\sigma(s_1)\sigma(u_1')$ cannot be the same as $\sigma(u_2)$. ∎

We say that a word $w'$ is obtained from a word $w$ by a *cyclic permutation* of letters if there exist words $u_1$ and $u_2$ such that $w = u_1u_2$ and $w' = u_2u_1$.

**Lemma 2.4** *Any word of length $2^n$ that occurs as a subword in $\xi$ can be obtained from one of the words $w_nb$, $w_nc$, and $w_nd$ by a cyclic permutation of letters.*

**Proof.** Since the infinite word $\xi = \xi_1\xi_2\dots$ is invariant under the substitution $\sigma$, it follows that $\xi = \sigma^n(\xi) = \sigma^n(\xi_1)\sigma^n(\xi_2)\dots$ Lemma 2.2 implies that $\xi_N = a$ if and only if $N$ is odd. Therefore $\xi = w_ns_1w_ns_2w_ns_3\dots$, where each $s_i \in \{b, c, d\}$. By Lemma 2.1, the length of the word $w_n$ is $2^n - 1$. It follows that any subword of length $2^n$ in $\xi$ is of the form $w_-lw_+$, where $l \in \{b, c, d\}$, $w_+$ is a beginning of $w_n$, and $w_-$ is an ending of $w_n$. Since the concatenated word $w_+w_-$ has the same length as $w_n$, it has to coincide with $w_n$. Then the word $w_-lw_+$ can be obtained from $w_nl$ by a cyclic permutation of letters. ∎

It turns out that the representation of the infinite word $\xi$ as $w_ns_1w_ns_2w_ns_3\dots$, where each $s_i \in \{b, c, d\}$, does not show all occurrences of $w_n$ as a subword in $\xi$. There are more occurrences, they overlap with the shown ones. As a result, it is not true that any occurrence of $w_n$ in $\xi$ is immediately followed by $bw_n$, $cw_n$, or $dw_n$. For example, some occurrences of $w_2 = aca$ are followed by $caba$. The next three lemmas explain what can follow and what can precede a particular occurrence of $w_n$.

**Lemma 2.5** *Any occurrence of the word $w_nl_n$ in $\xi$ is immediately followed by $w_nb$, $w_nc$, or $w_nd$ and, unless it is the beginning of $\xi$, immediately preceded by $w_nb$, $w_nc$, or $w_nd$.*

**Proof.** The proof is by induction on $n$. First consider the case $n = 1$. By Lemma 2.2, $\xi_N = a$ if and only if $N$ is odd. Therefore every occurrence of $w_1l_1 = ac$ in $\xi$ is immediately followed by $ab$, $ac$, or $ad$ and, unless it is the beginning of $\xi$, immediately preceded by $ab$, $ac$, or $ad$.

Now let $k \geq 1$ and assume the lemma holds for $n = k$. Suppose $\xi_{N+1}\xi_{N+2}\dots\xi_{N+M}$ is an occurrence of the word $w_{k+1}l_{k+1}$ in $\xi$. The first four letters of $w_{k+1}l_{k+1}$ are $acab$ so that $\xi_{N+4} = b$. Lemma 2.2 implies that $N + 4$ is divisible by 4. Besides, $M = 2^{k+1}$ due to Lemma 2.1. Hence $N$ and $M$ are both divisible by 4, i.e., $N = 4N'$ and $M = 4M'$ for some $M', N' \in \mathbb{Z}$. Since $\xi_{N+1}\xi_{N+2}\dots\xi_{N+M} = w_{k+1}l_{k+1} = \sigma(w_kl_k)$, it follows from Lemma 2.3 that $\xi_{2N'+1}\xi_{2N'+2}\dots\xi_{2N'+2M'}$ is an occurrence of $w_kl_k$. By the inductive assumption,

$\xi_{2N'+2M'+1}\xi_{2N'+2M'+2}\dots\xi_{2N'+4M'}$ is an occurrence of $w_kb$, $w_kc$, or $w_kd$ and, unless $N'=0$, we have $N'\geq M'$ and $\xi_{2N'-2M'+1}\xi_{2N'-2M'+2}\dots\xi_{2N'}$ is also an occurrence of $w_kb$, $w_kc$, or $w_kd$. Applying Lemma 2.3 two more times, we obtain that $\xi_{N+1}\xi_{N+2}\dots\xi_{N+M}$ is immediately followed by $\sigma(w_kb)=w_{k+1}d$, $\sigma(w_kc)=w_{k+1}b$, or $\sigma(w_kd)=w_{k+1}c$ and, unless $N=0$, immediately preceded by one of the same three words. This completes the induction step. ∎

**Lemma 2.6** *Any occurrence of the word $w_nl_{n+1}$ in $\xi$ is immediately followed by $w_nl_n$ and immediately preceded by another $w_nl_n$.*

**Proof.** The proof is by induction on $n$. First consider the case $n=1$. By Lemma 2.2, $\xi_N=a$ if $N$ is odd and $\xi_N=c$ if $N$ is even while not divisible by 4. It follows that every occurrence of $b$ or $d$ in $\xi$ is immediately followed and immediately preceded by $aca$. Therefore every occurrence of $w_1l_2=ab$ is immediately followed and preceded by $ac=w_1l_1$.

Now let $k\geq 1$ and assume the lemma holds for $n=k$. Suppose $\xi_{N+1}\xi_{N+2}\dots\xi_{N+M}$ is an occurrence of the word $w_{k+1}l_{k+2}$ in $\xi$. The first four letters of $w_{k+1}l_{k+2}$ are $acab$ (if $k>1$) or $acad$ (if $k=1$). In either case, Lemma 2.2 implies that $N+4$ is divisible by 4. Besides, $M=2^{k+1}$ due to Lemma 2.1. Hence $N$ and $M$ are both divisible by 4, i.e., $N=4N'$ and $M=4M'$ for some $M',N'\in\mathbb{Z}$. Since $\xi_{N+1}\xi_{N+2}\dots\xi_{N+M}=w_{k+1}l_{k+2}=\sigma(w_kl_{k+1})$, it follows from Lemma 2.3 that $\xi_{2N'+1}\xi_{2N'+2}\dots\xi_{2N'+2M'}$ is an occurrence of $w_kl_{k+1}$. By the inductive assumption, $N'\geq M'$ and

$$\xi_{2N'+2M'+1}\xi_{2N'+2M'+2}\dots\xi_{2N'+4M'}=\xi_{2N'-2M'+1}\xi_{2N'-2M'+2}\dots\xi_{2N'}=w_kl_k.$$

Applying Lemma 2.3 two more times, we obtain that $\xi_{N+1}\xi_{N+2}\dots\xi_{N+M}$ is immediately followed and immediately preceded by $\sigma(w_kl_k)=w_{k+1}l_{k+1}$. This completes the induction step. ∎

**Lemma 2.7** *Any occurrence of the word $w_{n+1}l_n=w_nl_nw_nl_n$ in $\xi$ is immediately followed and immediately preceded by the same word of length $2^{n+1}$, which can be either $w_nl_nw_nl_{n+1}$ or $w_nl_{n+1}w_nl_n$.*

**Proof.** The proof is by induction on $n$. First consider the case $n=1$. By Lemma 2.2, $\xi_N=a$ if $N$ is odd, $\xi_N=c$ if $N$ is even but not divisible by 4, and $\xi_N=b$ if $N$ is divisible by 4 but not by 8. Suppose $\xi_{M+1}\xi_{M+2}\xi_{M+3}\xi_{M+4}$ is an occurrence of $w_2l_1=acac$ in $\xi$. Then $M$ is even. Moreover, the one of the numbers $M+2$ and $M+4$ that is divisible by 4 must be divisible by 8 as well. In particular, $M\geq 4$. If $M+4$ is divisible by 8 then $\xi_{M+5}\xi_{M+6}\xi_{M+7}\xi_{M+8}=\xi_{M-3}\xi_{M-2}\xi_{M-1}\xi_M=acab=w_1l_1w_1l_2$. If $M+2$ is divisible by 8 then $\xi_{M+5}\xi_{M+6}\xi_{M+7}\xi_{M+8}=\xi_{M-3}\xi_{M-2}\xi_{M-1}\xi_M=abac=w_1l_2w_1l_1$.

Now let $k\geq 1$ and assume the lemma holds for $n=k$. Suppose $\xi_{N+1}\xi_{N+2}\dots\xi_{N+M}$ is an occurrence of the word $w_{k+2}l_{k+1}$ in $\xi$. The first four letters of $w_{k+2}l_{k+1}$ are $acab$ so that $\xi_{N+4}=b$. Lemma 2.2 implies that $N+4$ is divisible by 4. Besides, $M=2^{k+2}$ due to Lemma 2.1. Hence $N$ and $M$ are both divisible by 4, i.e., $N=4N'$ and $M=4M'$ for some $M',N'\in\mathbb{Z}$. Since $\xi_{N+1}\xi_{N+2}\dots\xi_{N+M}=w_{k+2}l_{k+1}=\sigma(w_{k+1}l_k)$, it follows from Lemma 2.3 that $\xi_{2N'+1}\xi_{2N'+2}\dots\xi_{2N'+2M'}$ is an occurrence of $w_{k+1}l_k$. By the inductive assumption, this occurrence is immediately followed and immediately preceded by the same word $u$ of even length, where $u = w_k l_k w_k l_{k+1}$ or $u = w_k l_{k+1} w_k l_k$. Applying Lemma 2.3 two more times, we obtain that $\xi_{N+1}\xi_{N+2}\ldots\xi_{N+M}$ is immediately followed and immediately preceded by $\sigma(u)$. Note that $\sigma(u) = \sigma(w_k l_k w_k l_{k+1}) = w_{k+1} l_{k+1} w_{k+1} l_{k+2}$ or $\sigma(u) = \sigma(w_k l_{k+1} w_k l_k) = w_{k+1} l_{k+2} w_{k+1} l_{k+1}$. The length of $\sigma(u)$ is $2^{k+2}$ due to Lemma 2.1. This completes the induction step. ∎

Next we are going to derive some properties of the cylinders in $\Omega$.

**Lemma 2.8** *Let $n \geq 2$. Then the cylinder $[.w_n]$ is disjoint from $T^N([.w_n])$ for $1 \leq N < 2^{n-1}$.*

**Proof.** Let $n \geq 2$ and suppose the cylinder $[.w_n]$ is not disjoint from $T^N([.w_n])$ for some $N \geq 1$. We need to show that $N \geq 2^{n-1}$. Take any element $\omega = \ldots\omega_{-2}\omega_{-1}\omega_0.\omega_1\omega_2\ldots$ of the intersection $[.w_n] \cap T^N([.w_n])$. Then $\omega$ and $T^{-N}(\omega)$ are both in $[.w_n]$. By construction of the infinite word $\xi = \xi_1\xi_2\ldots$, the word $w_n$ is a beginning of $\xi$. The length of $w_n$ is $2^n - 1$ due to Lemma 2.1. Since $\omega$ and $T^{-N}(\omega)$ belong to $[.w_n]$, it follows that $\omega_i = \omega_{i-N} = \xi_i$ for $1 \leq i \leq 2^n - 1$. As a consequence, $\xi_i = \xi_{i+N}$ whenever $1 \leq i < i + N \leq 2^n - 1$.

The integer $N$ is uniquely represented as $N = 2^k K$, where $k \geq 0$ and $K$ is odd. By Lemma 2.2, $\xi_N = l_k$ if $k \geq 1$ and $\xi_N = a$ if $k = 0$. By the same lemma, $\xi_{2N} = l_{k+1}$, which implies that $\xi_N \neq \xi_{2N} = \xi_{N+N}$. Then it follows from the above that $2N > 2^n - 1$. Since $N$ is an integer, this is equivalent to $2N \geq 2^n$ or $N \geq 2^{n-1}$. ∎

**Lemma 2.9** *$[l.w_n l_n] = [w_n l.w_n l_n w_n]$ and $[l_n.w_n l] = [w_n l_n.w_n l w_n]$ for all $l \in \{b, c, d\}$ and $n \geq 1$. Moreover, if $l \neq l_n$ then $[l.w_n l_n] = [w_n l_n w_n l.]$.*

**Proof.** By Lemma 2.5, any occurrence of the word $w_n l_n$ in $\xi$ is immediately followed by $w_n b$, $w_n c$, or $w_n d$ and, unless it is the beginning of $\xi$, immediately preceded by $w_n b$, $w_n c$, or $w_n d$. As a consequence, any occurrence of $l w_n l_n$ is immediately followed and preceded by $w_n$. This implies an equality of cylinders $[l.w_n l_n] = [w_n l.w_n l_n] = [w_n l.w_n l_n w_n]$. In the case $l = l_n$, we are done. In the other cases, two more equalities are to be derived.

Next consider the case $l = l_{n+1}$. By Lemma 2.6, any occurrence of the word $w_n l_{n+1}$ in $\xi$ is immediately followed and preceded by $w_n l_n$. Therefore any occurrence of $l_n w_n l$ is immediately followed and preceded by $w_n$ so that $[l_n.w_n l] = [w_n l_n.w_n l w_n]$. Besides, this implies that $[w_n l_n w_n l.] = [w_n l.] = [w_n l.w_n l_n]$. We already know that $[w_n l.w_n l_n] = [l.w_n l_n]$.

Now consider the case when $l \neq l_n$, $l \neq l_{n+1}$, and $n \geq 2$. In this case, $l = l_{n-1}$. By Lemma 2.7, any occurrence of the word $w_n l_{n-1} = w_{n-1} l_{n-1} w_{n-1} l_{n-1}$ in $\xi$ is immediately followed and preceded by the same word of length $2^n$, which can be either $w_{n-1} l_{n-1} w_{n-1} l_n = w_n l_n$ or $w_{n-1} l_n w_{n-1} l_{n-1}$. Therefore any occurrence of $l_n w_n l$ is immediately followed and preceded by $w_n$ so that $[l_n.w_n l] = [w_n l_n.w_n l w_n]$. Another consequence is that $[w_n l_n w_n l.] = [w_n l_n w_n l.w_n l_n] = [w_n l.w_n l_n]$ (unlike the previous case, this does not equal $[w_n l.]$). We already know that $[w_n l.w_n l_n] = [l.w_n l_n]$.

It remains to consider the case when $l \neq l_n$, $l \neq l_{n+1}$, and $n = 1$. In this case, $w_n = a$, $l_n = c$, and $l = d$. As already observed in the proof of Lemma 2.6, every occurrence of the letter $d$ in $\xi$ is immediately followed and preceded by $aca$. Therefore $[c.ad] = [ac.ada]$ and $[d.ac] = [acad.aca] = [acad.]$. ∎

The following lemma is crucial for the proof of Theorem 1.10.

**Lemma 2.10** *If $C$ is a nonempty cylinder of dimension $2^n$, then $C = T^N([.w_n l])$ for some $l \in \{b, c, d\}$ and $N \in \mathbb{Z}$.*

**Proof.** Let $C$ be a nonempty cylinder of dimension $2^n$. We have $C = [w_-.w_+]$ for some words $w_-$ and $w_+$ such that the concatenated word $w = w_- w_+$ has length $2^n$. Since $C$ is a nonempty set, the word $w$ must occur as a subword in the infinite word $\xi$. By Lemma 2.4, $w$ can be obtained from a word $w_n l$, $l \in \{b, c, d\}$, by a cyclic permutation of letters. We are going to show that $[.w] = T^N([.w_n l])$ for some $N \in \mathbb{Z}$. Then $C = T^M([.w]) = T^{M+N}([.w_n l])$, where $M$ is the length of $w_-$.

First consider the case $n = 1$. In this case, $w = w_1 l = al$ or $w = la$. By Lemma 2.2, $\xi_N = a$ if and only if $N$ is odd. Hence every occurrence of the letter $l$ in $\xi$ is immediately followed and preceded by $a$. Therefore $[.la] = [.l] = [a.l] = T([.al])$.

Now assume $n \geq 2$. In this case, $w_n = w_{n-1} l_{n-1} w_{n-1}$ due to Lemma 2.1. Let $u_1$ and $u_2$ be words such that $w_n l = u_1 u_2$ and $w = u_2 u_1$. If $u_2$ is longer than $u_1$, then $l_{n-1} w_{n-1} l$ is an ending of $u_2$, i.e., $u_2 = u_2' l_{n-1} w_{n-1} l$ for some word $u_2'$. Clearly, $w = u_2' l_{n-1} w_{n-1} l u_1$ and $u_1 u_2' = w_{n-1}$. If $u_2$ is not longer than $u_1$ and not empty, we have $u_1 = w_{n-1} l_{n-1} u_1'$ and $u_2 = u_2' l$ for some words $u_1'$ and $u_2'$. Then $w = u_2' l w_{n-1} l_{n-1} u_1'$ and $u_1' u_2' = w_{n-1}$. Finally, if $u_2$ is empty, then $w = w_n l = w_{n-1} l_{n-1} w_{n-1} l = u_2' l_{n-1} w_{n-1} l u_1'$, where $u_1'$ is the empty word and $u_2' = w_{n-1}$.

By the above the word $w$ can be represented as $u_2' l_{n-1} w_{n-1} l u_1'$ or $u_2' l w_{n-1} l_{n-1} u_1'$, where the words $u_1'$ and $u_2'$ satisfy $u_1' u_2' = w_{n-1}$. Note that both representations are the same if $l = l_{n-1}$ (also, in this case there are two different choices for the pair $u_1, u_2$). First assume that $w = u_2' l_{n-1} w_{n-1} l u_1'$. Since $u_1'$ is a beginning of $w_{n-1}$ and $u_2'$ is an ending of $w_{n-1}$, it follows that $[w_{n-1} l_{n-1}.w_{n-1} l w_{n-1}] \subset [u_2' l_{n-1}.w_{n-1} l u_1'] \subset [l_{n-1}.w_{n-1} l]$. Similarly, $[w_{n-1} l_{n-1}.w_{n-1} l w_{n-1}] \subset [w_{n-1} l_{n-1}.w_{n-1} l] \subset [l_{n-1}.w_{n-1} l]$. Since $[w_{n-1} l_{n-1}.w_{n-1} l w_{n-1}] = [l_{n-1}.w_{n-1} l]$ due to Lemma 2.9, we obtain that $[u_2' l_{n-1}.w_{n-1} l u_1'] = [w_{n-1} l_{n-1}.w_{n-1} l]$. The latter equality can be rewritten as $T^{N_1}([.w]) = T^{N_2}([.w_n l])$, where $N_1$ is the length of $u_2' l_{n-1}$ and $N_2$ is the length of $w_{n-1} l_{n-1}$ ($N_2 = 2^{n-1}$). Then $[.w] = T^{N_2 - N_1}([.w_n l])$.

Now assume that $l \neq l_{n-1}$ and $w = u_2' l w_{n-1} l_{n-1} u_1'$. Just like in the previous case, we obtain that $[w_{n-1} l.w_{n-1} l_{n-1} w_{n-1}] \subset [u_2' l.w_{n-1} l_{n-1} u_1'] \subset [l.w_{n-1} l_{n-1}]$. By Lemma 2.9, $[w_{n-1} l.w_{n-1} l_{n-1} w_{n-1}] = [l.w_{n-1} l_{n-1}] = [w_{n-1} l_{n-1} w_{n-1} l.]$. It follows that $[u_2' l.w_{n-1} l_{n-1} u_1'] = [w_{n-1} l_{n-1} w_{n-1} l.]$. The latter equality can be rewritten as $T^{N_1}([.w]) = T^{N_2}([.w_n l])$, where $N_1$ is the length of $u_2' l$ and $N_2$ is the length of $w_n l$ ($N_2 = 2^n$). Then $[.w] = T^{N_2 - N_1}([.w_n l])$. ∎

The next three lemmas establish relations between cylinders of dimension $2^n$ and cylinders of dimension $2^{n+1}$. We shall use $\sqcup$ to denote disjoint unions. Namely, $U = U_1 \sqcup U_2 \sqcup \ldots \sqcup U_k$ means that $U = U_1 \cup U_2 \cup \ldots \cup U_k$ and the sets $U_1, U_2, \ldots, U_k$ are pairwise disjoint.

**Lemma 2.11** $[.w_n l_n] = [.w_{n+1} b] \sqcup [.w_{n+1} c] \sqcup [.w_{n+1} d]$ *for all* $n \geq 1$.

**Proof.** The cylinders $[.w_{n+1} b]$, $[.w_{n+1} c]$, and $[.w_{n+1} d]$ are clearly disjoint. Since $w_{n+1} = w_n l_n w_n$ (due to Lemma 2.1), each of them is contained in $[.w_n l_n]$. Lemma 2.5 implies that the union of the three cylinders is exactly $[.w_n l_n]$. ∎

**Lemma 2.12** $[.w_n l_{n+1}] = T^{2^n}([.w_{n+1} l_{n+1}])$ *for all* $n \geq 1$.

**Proof.** Lemma 2.6 implies that $[.w_n l_{n+1}] = [w_n l_n.w_n l_{n+1}]$. Since the length of the word $w_n l_n$ is $2^n$, we obtain that $[.w_n l_{n+1}] = T^{2^n}([.w_n l_n w_n l_{n+1}])$. It remains to notice that $w_n l_n w_n = w_{n+1}$. ∎

**Lemma 2.13** $[.w_{n+1} l_n] = T^{2^{n+1}}([.w_{n+2} l_n]) \sqcup T^{3\cdot 2^n}([.w_{n+2} l_n])$ *for all* $n \geq 1$.

**Proof.** Lemma 2.7 implies that the cylinder $[.w_{n+1} l_n] = [.w_n l_n w_n l_n]$ is the union of cylinders $C_1 = [w_n l_n w_n l_{n+1}.w_n l_n w_n l_n]$ and $C_2 = [w_n l_{n+1} w_n l_n.w_n l_n w_n l_n]$, which are disjoint since $l_{n+1}$ is always different from $l_n$. Lemma 2.6 further implies that $C_2 = [w_n l_n w_n l_{n+1} w_n l_n.w_n l_n w_n l_n]$. Then it follows from Lemma 2.7 that $C_2 = [w_n l_n w_n l_{n+1} w_n l_n.w_n l_n]$.

Notice that $w_n l_n w_n l_{n+1} w_n l_n w_n l_n = w_{n+1} l_{n+1} w_{n+1} l_n = w_{n+2} l_n$. Since the word $w_n l_n w_n l_{n+1}$ has length $2^{n+1}$ and the word $w_n l_n w_n l_{n+1} w_n l_n$ has length $3 \cdot 2^n$, we obtain that $C_1 = T^{2^{n+1}}([.w_{n+2} l_n])$ and $C_2 = T^{3\cdot 2^n}([.w_{n+2} l_n])$. ∎

# 3 General topological full group

We proceed to the study of the topological full group $[[T]]$. Let us begin with some general properties of transformations $\Psi_{U,M,N}$ that hold for any homeomorphism $T : X \to X$ of a Cantor set $X$ onto itself.

**Lemma 3.1** *If* $\Psi_{U,M,N}$ *is well defined for a clopen set* $U$ *and integers* $M, N$, $M < N$, *then* $\Psi_{T^K(U),M+J,N+J}$ *is well defined for any* $J, K \in \mathbb{Z}$ *and* $\Psi_{T^K(U),M+J,N+J} = T^{J+K}\Psi_{U,M,N}T^{-J-K}$.

**Proof.** Since $\Psi_{U,M,N}$ is well defined, the sets $T^M(U), T^{M+1}(U), \dots, T^N(U)$ are pairwise disjoint. Since $T$ is an invertible transformation, it follows that for any $J \in \mathbb{Z}$ the sets $T^{M+J}(U), T^{M+J+1}(U), \dots, T^{N+J}(U)$ are also pairwise disjoint. Hence $\Psi_{U,M+J,N+J}$ is defined as well. Suppose $x \in X$ and let $y = T^{-J}(x)$. Then $\Psi_{U,M+J,N+J}(x) = T^n(x)$ for a specific $n$ (which can be 0, 1, or $M-N$) if and only if $\Psi_{U,M,N}(y) = T^n(y)$. It follows that $\Psi_{U,M+J,N+J} = T^J\Psi_{U,M,N}T^{-J}$.

Given $K \in \mathbb{Z}$, let $V = T^K(U)$. Then $V$ is a clopen set and $T^i(V) = T^{i+K}(U)$ for all $i \in \mathbb{Z}$. It follows that $\Psi_{V,M',N'} = \Psi_{U,M'+K,N'+K}$ whenever one of these transformations is defined. In particular, $\Psi_{V,M+J,N+J} = \Psi_{U,M+J+K,N+J+K}$ for all $J \in \mathbb{Z}$. By the above, $\Psi_{U,M+J+K,N+J+K} = T^{J+K}\Psi_{U,M,N}T^{-J-K}$. ∎

**Lemma 3.2** *Suppose* $\Psi_{U,M,N}$ *is well defined and* $U = U_1 \sqcup U_2 \sqcup \ldots \sqcup U_k$, *where* $U_1, U_2, \dots, U_k$ *are clopen sets. Then transformations* $\Psi_{U_i,M,N}$, $1 \leq i \leq k$ *are also well defined, they commute with one another, and* $\Psi_{U,M,N} = \Psi_{U_1,M,N}\Psi_{U_2,M,N}\dots\Psi_{U_k,M,N}$.

**Proof.** Since $\Psi_{U,M,N}$ is well defined, the sets $T^M(U), T^{M+1}(U), \dots, T^N(U)$ are pairwise disjoint. Since each $U_i$ is a subset of $U$, the sets $T^M(U_i), T^{M+1}(U), \dots, T^N(U_i)$ are also pairwise disjoint. Hence $\Psi_{U_i,M,N}$ is defined as well. The transformation $\Psi_{U_i,M,N}$ coincides with $\Psi_{U,M,N}$ on the set $\widetilde{U}_i = T^M(U_i) \cup T^{M+1}(U_i) \cup \cdots \cup T^N(U_i)$ and with the identity map anywhere else. Since $U = U_1 \sqcup U_2 \sqcup \ldots \sqcup U_k$, it follows that $T^J(U) = T^J(U_1) \sqcup T^J(U_2) \sqcup \ldots \sqcup T^J(U_k)$ for all $J \in \mathbb{Z}$. As a consequence, $T^M(U) \cup T^{M+1}(U) \cup \cdots \cup T^N(U) = \widetilde{U}_1 \sqcup \widetilde{U}_2 \sqcup \ldots \sqcup \widetilde{U}_k$. This implies that transformations $\Psi_{U_1,M,N}, \Psi_{U_2,M,N}, \ldots, \Psi_{U_k,M,N}$ commute with one another and $\Psi_{U_1,M,N}\Psi_{U_2,M,N} \ldots \Psi_{U_k,M,N} = \Psi_{U,M,N}$. ∎

**Lemma 3.3** *If $\Psi_{U,M,N}$ is well defined and $N - M \geq 2$, then $\Psi_{U,M,N} = \Psi_{U,M,K}\Psi_{U,K,N}$ for any $K$, $M < K < N$.*

**Proof.** Since $\Psi_{U,M,N}$ is well defined, the sets $T^M(U), T^{M+1}(U), \ldots, T^N(U)$ are pairwise disjoint. It follows that transformations $\Psi_{U,M,K}$ and $\Psi_{U,K,N}$ are well defined for any $K$, $M < K < N$. We need to show that $\Psi_{U,M,N}(x) = \Psi_{U,M,K}(\Psi_{U,K,N}(x))$ for all $x \in X$. First consider the case $x \in T^i(U)$, where $M \leq i \leq K - 1$. Then $x \notin T^j(U)$ for $K \leq j \leq N$. Hence $x$ is fixed by $\Psi_{U,K,N}$. Consequently, $\Psi_{U,M,K}(\Psi_{U,K,N}(x)) = \Psi_{U,M,K}(x) = T(x)$, which coincides with $\Psi_{U,M,N}(x)$.

Next consider the case $x \in T^i(U)$, where $K \leq i \leq N-1$. In this case, $\Psi_{U,K,N}(x) = T(x)$. Since $T(x) \in T^{i+1}(U)$ and $K+1 \leq i+1 \leq N$, it follows that $T(x) \notin T^j(U)$ for $M \leq j \leq K$. Hence $T(x)$ is fixed by $\Psi_{U,M,K}$ so that $\Psi_{U,M,K}(\Psi_{U,K,N}(x)) = T(x) = \Psi_{U,M,N}(x)$.

Now consider the case $x \in T^N(U)$. In this case, $\Psi_{U,K,N}(x) = T^{K-N}(x)$, which belongs to $T^K(U)$. Then $\Psi_{U,M,K}(\Psi_{U,K,N}(x)) = \Psi_{U,M,K}(T^{K-N}(x)) = T^{M-K}(T^{K-N}(x)) = T^{M-N}(x)$, which coincides with $\Psi_{U,M,N}(x)$.

Finally, if $x \notin T^i(U)$ for all $i$, $M \leq i \leq N$, then $x$ is fixed by all three transformations. In particular, $\Psi_{U,M,K}(\Psi_{U,K,N}(x)) = x = \Psi_{U,M,N}(x)$. ∎

**Lemma 3.4** *Suppose $\Psi_{U,M,K}$ and $\Psi_{V,K,N}$ are well defined. If $T^i(V) \cap U = \emptyset$ for $1 \leq i \leq N - M$, then $\Psi_{V,K,N}\Psi^{-1}_{U,M,K}\Psi^{-1}_{V,K,N}\Psi_{U,M,K} = \Psi_{U\cap V,K-1,K+1}$.*

**Proof.** Since $\Psi_{U,M,K}$ is well defined, the sets $T^M(U), T^{M+1}(U), \ldots, T^K(U)$ are pairwise disjoint. Since $\Psi_{V,K,N}$ is well defined, the sets $T^K(V), T^{K+1}(V), \ldots, T^N(V)$ are pairwise disjoint. Further, $T^i(U) \cap T^j(V) = T^i(U \cap T^{j-i}(V))$ for all $i, j \in \mathbb{Z}$. Therefore $T^i(U)$ is disjoint from $T^j(V)$ whenever $1 \leq j - i \leq N - M$. In particular, the two sets are disjoint if $M \leq i \leq K \leq j \leq N$ and at least one of the numbers $i$ and $j$ is different from $K$. It follows that sets $T^M(U), T^{M+1}(U), \ldots, T^{K-1}(U), T^K(U) \cup T^K(V) = T^K(U \cup V)$, $T^{K+1}(V), \ldots, T^{N-1}(V), T^N(V)$ are pairwise disjoint.

Let $W = U \cap V$, $Y = U \setminus W$, and $Z = V \setminus W$. Then $W$, $Y$, and $Z$ are clopen sets. We have $U = W \sqcup Y$, $V = W \sqcup Z$, and $U \cup V = W \sqcup Y \sqcup Z$. By Lemma 3.2, $\Psi_{U,M,K} = \Psi_{W,M,K}\Psi_{Y,M,K}$ and $\Psi_{V,K,N} = \Psi_{W,K,N}\Psi_{Z,K,N}$. The transformation $\Psi_{Y,M,K}$ moves points only within the set $\widetilde{Y} = T^M(Y) \cup T^{M+1}(Y) \cup \cdots \cup T^K(Y)$. Likewise, $\Psi_{Z,K,N}$ moves points only within the set $\widetilde{Z} = T^K(Z) \cup T^{K+1}(Z) \cup \cdots \cup T^N(Z)$. The transformations $\Psi_{W,M,K}$ and $\Psi_{W,K,N}$ do not move points outside of the set $\widetilde{W} = T^M(W) \cup T^{M+1}(W) \cup \cdots \cup T^N(W)$. Note that $T^i(U) = T^i(W) \sqcup T^i(Y)$ for $M \leq i \leq K-1$, $T^i(V) = T^i(W) \sqcup T^i(Z)$ for $K+1 \leq i \leq N$, and $T^K(U \cup V) = T^K(W) \sqcup T^K(Y) \sqcup T^K(Z)$. It follows that the sets $\widetilde{W}$, $\widetilde{Y}$, and $\widetilde{Z}$ are pairwise disjoint. This implies that the transformations $\Psi_{Y,M,K}$ and $\Psi_{Z,K,N}$ commute with $\Psi_{W,M,K}$, $\Psi_{W,K,N}$, and with each other. Then

$$\Psi_{V,K,N}\Psi^{-1}_{U,M,K}\Psi^{-1}_{V,K,N}\Psi_{U,M,K} =$$

$$\begin{aligned}
&= (\Psi_{W,K,N}\Psi_{Z,K,N})(\Psi_{W,M,K}\Psi_{Y,M,K})^{-1}(\Psi_{W,K,N}\Psi_{Z,K,N})^{-1}(\Psi_{W,M,K}\Psi_{Y,M,K})\\
&= \Psi_{W,K,N}\Psi_{Z,K,N}\Psi^{-1}_{Y,M,K}\Psi^{-1}_{W,M,K}\Psi^{-1}_{Z,K,N}\Psi^{-1}_{W,K,N}\Psi_{W,M,K}\Psi_{Y,M,K}\\
&= \Psi_{W,K,N}\Psi^{-1}_{W,M,K}(\Psi_{Z,K,N}\Psi^{-1}_{Y,M,K}\Psi^{-1}_{Z,K,N}\Psi_{Y,M,K})\Psi^{-1}_{W,K,N}\Psi_{W,M,K}\\
&= \Psi_{W,K,N}\Psi^{-1}_{W,M,K}\Psi^{-1}_{W,K,N}\Psi_{W,M,K}.
\end{aligned}$$

Let $L = \Psi_{W,M,K-1}$ if $M < K-1$ and let $L$ be the identity map otherwise. Let $R = \Psi_{W,K+1,N}$ if $K+1 < N$ and let $R$ be the identity map otherwise. It follows from Lemma 3.3 that $\Psi_{W,M,K} = L\Psi_{W,K-1,K}$ and $\Psi_{W,K,N} = \Psi_{W,K,K+1}R$. The transformation $L$ fixes all points in the set $T^K(W) \cup T^{K+1}(W) \cup \cdots \cup T^N(W)$, which implies that $L$ commutes with $\Psi_{W,K,K+1}$ and $R$. Similarly, $R$ fixes all points in the set $T^M(W) \cup T^{M+1}(W) \cup \cdots \cup T^K(W)$, which implies that $R$ commutes with $\Psi_{W,K-1,K}$ and $L$. Then

$$\Psi_{W,K,N}\Psi^{-1}_{W,M,K}\Psi^{-1}_{W,K,N}\Psi_{W,M,K} = (\Psi_{W,K,K+1}R)(L\Psi_{W,K-1,K})^{-1}(\Psi_{W,K,K+1}R)^{-1}(L\Psi_{W,K-1,K})$$

$$\begin{aligned}
&= \Psi_{W,K,K+1}R\Psi^{-1}_{W,K-1,K}L^{-1}R^{-1}\Psi^{-1}_{W,K,K+1}L\Psi_{W,K-1,K}\\
&= \Psi_{W,K,K+1}\Psi^{-1}_{W,K-1,K}(RL^{-1}R^{-1}L)\Psi^{-1}_{W,K,K+1}\Psi_{W,K-1,K}\\
&= \Psi_{W,K,K+1}\Psi^{-1}_{W,K-1,K}\Psi^{-1}_{W,K,K+1}\Psi_{W,K-1,K}.
\end{aligned}$$

Since $\Psi_{W,K-1,K}$ and $\Psi_{W,K,K+1}$ are involutions, we obtain that

$$\Psi_{W,K,K+1}\Psi^{-1}_{W,K-1,K}\Psi^{-1}_{W,K,K+1}\Psi_{W,K-1,K} = (\Psi_{W,K,K+1}\Psi_{W,K-1,K})^2 = (\Psi_{W,K-1,K}\Psi_{W,K,K+1})^{-2}.$$

It follows from the above that sets $T^M(W), \dots, T^{K-1}(W), T^K(W), T^{K+1}(W), \dots, T^N(W)$ are pairwise disjoint. In particular, the transformation $\Psi_{W,K-1,K+1}$ is well defined. We have $\Psi_{W,K-1,K}\Psi_{W,K,K+1} = \Psi_{W,K-1,K+1}$ due to Lemma 3.3 and $\Psi^{-2}_{W,K-1,K+1} = \Psi_{W,K-1,K+1}$ since $\Psi_{W,K-1,K+1}$ has order 3. ∎

# 4 Topological full group of the substitution subshift

Now we restrict our attention to the substitution subshift $T : \Omega \to \Omega$. Let $G$ be the subgroup of $[[T]]$ generated by transformations $T$, $\delta_{[.b]}$, $\delta_{[.d]}$, and $\delta_{[.acacac]}$. For any $n \geq 1$ let $G_n$ be the subgroup of $[[T]]$ generated by $\delta_{[.w_nb]}$, $\delta_{[.w_nc]}$, $\delta_{[.w_nd]}$, and $T$.

**Lemma 4.1** *$G_3 = G$.*

**Proof.** First we show that the group $G_3$ contains $\delta_{[.w_2b]}$. By Lemma 2.11, $[.w_2b] = [.w_3b] \sqcup [.w_3c] \sqcup [.w_3d]$. Then Lemma 3.2 implies that $\delta_{[.w_2b]} = \delta_{[.w_3b]}\delta_{[.w_3c]}\delta_{[.w_3d]}$.

By Lemma 2.2, $\xi_i = a$ if $i$ is odd, $\xi_i = c$ if $i$ is even but not divisible by 4, and $\xi_i = b$ if $i$ is divisible by 4 but not by 8. It follows that every occurrence of the letter $b$ in $\xi$ is immediately preceded by $aca$ while every occurrence of $d$ is preceded by $acabaca$. As a consequence, $[.b] = [aca.b] = T^3([.w_2b])$ and $[.d] = [acabaca.d] = T^7([.w_3d])$. Besides, Lemma 2.7 implies that $[.acacac] = [acab.acacacab] = [acab.acac] = T^4([.w_3c])$. By Lemma 3.1, $\delta_{[.b]} = T^3\delta_{[.w_2b]}T^{-3}$, $\delta_{[.d]} = T^7\delta_{[.w_3d]}T^{-7}$, and $\delta_{[.acacac]} = T^4\delta_{[.w_3c]}T^{-4}$. Therefore all generators of the group $G$ belong to $G_3$ so that $G \subset G_3$.

Conversely, it follows from the above that $\delta_{[.w_2b]} = T^{-3}\delta_{[.b]}T^3$, $\delta_{[.w_3c]} = T^{-4}\delta_{[.acacac]}T^4$, $\delta_{[.w_3d]} = T^{-7}\delta_{[.d]}T^7$, and

$$\begin{aligned} \delta_{[.w_3b]} &= \delta_{[.w_2b]}\delta^{-1}_{[.w_3d]}\delta^{-1}_{[.w_3c]} = \delta_{[.w_2b]}\delta_{[.w_3d]}\delta_{[.w_3c]} \\ &= (T^{-3}\delta_{[.b]}T^3)(T^{-7}\delta_{[.d]}T^7)(T^{-4}\delta_{[.acacac]}T^4) = T^{-3}\delta_{[.b]}T^{-4}\delta_{[.d]}T^3\delta_{[.acacac]}T^4. \end{aligned}$$

Therefore all generators of the group $G_3$ belong to $G$ so that $G_3 \subset G$. ∎

As a follow-up to the previous proof, let us derive the formulas for $\delta_{[.a]}$ and $\delta_{[.c]}$. We begin with some auxiliary formulas. By Lemma 2.13, $[.acac] = T^4([.w_3c]) \sqcup T^6([.w_3c])$. Then Lemmas 3.1 and 3.2 imply that

$$\delta_{[.acac]} = (T^4\delta_{[.w_3c]}T^{-4})(T^6\delta_{[.w_3c]}T^{-6}) = T^4\delta_{[.w_3c]}T^2\delta_{[.w_3c]}T^{-6}.$$

Since $\delta_{[.w_3c]} = T^{-4}\delta_{[.acacac]}T^4$, we obtain that $\delta_{[.acac]} = \delta_{[.acacac]}T^2\delta_{[.acacac]}T^{-2}$. Further, $[.acad] = T^4([.w_3d])$ due to Lemma 2.12. Hence $\delta_{[.acad]} = T^4\delta_{[.w_3d]}T^{-4} = T^4(T^{-7}\delta_{[.d]}T^7)T^{-4} = T^{-3}\delta_{[.d]}T^3$. Next, $[.ac] = [.acab] \sqcup [.acac] \sqcup [.acad]$ due to Lemma 2.12. By Lemma 3.2,

$$\begin{aligned} \delta_{[.ac]} &= \delta_{[.acab]}\delta_{[.acad]}\delta_{[.acac]} = (T^{-3}\delta_{[.b]}T^3)(T^{-3}\delta_{[.d]}T^3)(\delta_{[.acacac]}T^2\delta_{[.acacac]}T^{-2}) \\ &= T^{-3}\delta_{[.b]}\delta_{[.d]}T^3\delta_{[.acacac]}T^2\delta_{[.acacac]}T^{-2}. \end{aligned}$$

Finally, $[.c] = [a.c] = T([.ac])$ so that

$$\delta_{[.c]} = T\delta_{[.ac]}T^{-1} = T^{-2}\delta_{[.b]}\delta_{[.d]}T^3\delta_{[.acacac]}T^2\delta_{[.acacac]}T^{-3}.$$

Since $[.a] = T^{-1}([a.])$ and $[a.] = [.b] \sqcup [.c] \sqcup [.d]$, it follows from Lemma 3.2 that $\delta_{[a.]} = \delta_{[.b]}\delta_{[.d]}\delta_{[.c]}$ and then from Lemma 3.1 that

$$\delta_{[.a]} = T^{-1}\delta_{[a.]}T = T^{-1}\delta_{[.b]}\delta_{[.d]}\delta_{[.c]}T = T^{-1}\delta_{[.b]}\delta_{[.d]}T^{-2}\delta_{[.b]}\delta_{[.d]}T^3\delta_{[.acacac]}T^2\delta_{[.acacac]}T^{-2}.$$

**Lemma 4.2** *Any given transformation of the form $\delta_U$ is contained in the group $G_n$ for $n$ large enough.*

**Proof.** If $U$ is an empty set, then $\delta_U$ is the identity map. Now suppose $U \subset \Omega$ is a nonempty clopen set. Then there exists $n_0 \geq 1$ such that for any $n \geq n_0$ the set $U$ can be represented as a union $U = C_1 \cup C_2 \cup \cdots \cup C_s$, where each $C_i$ is of the form $[u.w]$ for some words $u, w$ of length $2^{n-1}$. We can assume that the cylinders $C_1, C_2, \dots, C_s$ are nonempty and distinct. Then $U = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_s$. If $\delta_U$ is well defined, then $\delta_U = \delta_{C_1}\delta_{C_2}\dots\delta_{C_s}$ due to Lemma 3.2. Since each $C_i$ is a nonempty cylinder of dimension $2^n$, Lemma 2.10 implies that $C_i = T^N([.w_nl])$ for some $l \in \{b, c, d\}$ and $N \in \mathbb{Z}$. Then $\delta_{C_i} = T^N\delta_{[.w_nl]}T^{-N}$ due to Lemma 3.1. In particular, each $\delta_{C_i}$ belongs to the group $G_n$. It follows that $\delta_U \in G_n$ as well. ∎

For any $n \geq 3$ let $H_n$ be the subgroup of $[[T]]$ generated by $\tau_{[.w_n b]}$, $\tau_{[.w_n c]}$, $\tau_{[.w_n d]}$, and $T$. The restriction $n \geq 3$ is necessary since $\tau_{[.ac]}$ and $\tau_{[.acac]}$ are not defined. If $n \geq 3$ then the cylinders $[.w_n b]$, $[.w_n c]$, and $[.w_n d]$ are contained in $[.acab]$. Since sets $[.acab]$, $T([.acab]) = [a.cab]$, and $T^2([.acab]) = [ac.ab]$ are disjoint from one another, the transformation $\tau_{[.acab]}$ is well defined and so are the generators of the group $H_n$.

**Lemma 4.3** *$\delta_{[.w_{n+1}l_n]} \in H_{n+2}$ for all $n \geq 1$.*

**Proof.** By Lemma 2.13, $[.w_{n+1}l_n] = T^{N_n}(U_n) \sqcup T^{M_n}(U_n)$, where $U_n = [.w_{n+2}l_n]$, $N_n = 2^{n+1}$, and $M_n = 3 \cdot 2^n$. Note that $M_n - N_n \geq 2$. It follows from Lemmas 3.1 and 3.2 that $\delta_{[.w_{n+1}l_n]} = \Psi_{U_n,N_n,N_n+1}\Psi_{U_n,M_n,M_n+1}$. Since $\Psi_{U_n,N,N+1}$ is an involution for all $N \in \mathbb{Z}$, we obtain

$$\delta_{[.w_{n+1}l_n]} = \Psi_{U_n,N_n,N_n+1}\Psi^2_{U_n,N_n+1,N_n+2}\Psi^2_{U_n,N_n+2,N_n+3}\dots\Psi^2_{U_n,M_n-1,M_n}\Psi_{U_n,M_n,M_n+1}$$

$$= (\Psi_{U_n,N_n,N_n+1}\Psi_{U_n,N_n+1,N_n+2})(\Psi_{U_n,N_n+1,N_n+2}\Psi_{U_n,N_n+2,N_n+3})\dots(\Psi_{U_n,M_n-1,M_n}\Psi_{U_n,M_n,M_n+1}).$$

Since $\tau_{U_n}$ is well defined, it follows from Lemma 3.1 that $\Psi_{U_n,N,N+2}$ is well defined for any $N \in \mathbb{Z}$. By Lemma 3.3, $\Psi_{U_n,N,N+2} = \Psi_{U_n,N,N+1}\Psi_{U_n,N+1,N+2}$ for all $N \in \mathbb{Z}$. Therefore

$$\delta_{[.w_{n+1}l_n]} = \Psi_{U_n,N_n,N_n+2}\Psi_{U_n,N_n+1,N_n+3}\dots\Psi_{U_n,M_n-2,M_n}.$$

By Lemma 3.1, $\Psi_{U_n,N,N+2} = T^N\tau_{U_n}T^{-N}$ for all $N \in \mathbb{Z}$. It follows by induction that

$$\Psi_{U_n,N,N+2}\Psi_{U_n,N+1,N+3}\dots\Psi_{U_n,N+K-1,N+K+1} = T^N(\tau_{U_n}T)^K\,T^{-N-K}$$

for all $N \in \mathbb{Z}$ and $K \geq 1$. In the case $N = N_n$, $K = M_n - N_n - 1$, we obtain that

$$\delta_{[.w_{n+1}l_n]} = T^{2^{n+1}}(\tau_{[.w_{n+2}l_n]}T)^{2^n-1}\,T^{1-3\cdot 2^n},$$

which belongs to the group $H_{n+2}$. ■

**Lemma 4.4** *$H_n = H_4$ for all $n > 4$.*

**Proof.** Let us fix an arbitrary $n \geq 4$. First we are going to show that $\tau_{[.w_{n+1}l]} \in H_n$ for each $l \in \{b, c, d\}$ (so that $H_{n+1} \subset H_n$). Let $U = [w_n l_n.]$ and $V_l = [.w_n l]$. By Lemma 2.1, $w_n$ has length $2^n - 1$ and $w_{n+1} = w_n l_n w_n$. It follows that $U = T^{2^n}([.w_n l_n])$ and $U \cap V_l = [w_n l_n.w_n l] = T^{2^n}([.w_{n+1}l])$. By Lemma 2.11, the cylinder $[.w_n l_n]$ is the union of $[.w_{n+1}b]$, $[.w_{n+1}c]$, and $[.w_{n+1}d]$. Therefore $U$ is the union of $T^{2^n}([.w_{n+1}b]) = [w_n l_n.w_n b]$, $T^{2^n}([.w_{n+1}c]) = [w_n l_n.w_n c]$, and $T^{2^n}([.w_{n+1}d]) = [w_n l_n.w_n d]$. As a consequence, the cylinder $U$ is contained in $[.w_n]$. Clearly, $V_l \subset [.w_n]$ as well.

By Lemma 2.8, the cylinder $[.w_n]$ is disjoint from $T^N([.w_n])$ for $1 \leq N < 2^{n-1}$. In particular, it is disjoint from $T([.w_n])$, $T^2([.w_n])$, $T^3([.w_n])$, and $T^4([.w_n])$. Since $U$ and $V_l$ are subsets of $[.w_n]$, the cylinder $U$ is disjoint from $T(V_l)$, $T^2(V_l)$, $T^3(V_l)$, and $T^4(V_l)$. Then it follows from Lemma 3.4 that

$$\Psi_{V_l,0,2}\Psi^{-1}_{U,-2,0}\Psi^{-1}_{V_l,0,2}\Psi_{U,-2,0} = \Psi_{U\cap V_l,-1,1}.$$

By Lemma 3.1, $\Psi_{U\cap V_l,-1,1} = T^{2^n-1}\tau_{[.w_{n+1}l]}T^{1-2^n}$ and $\Psi_{U,-2,0} = T^{2^n-2}\tau_{[.w_nl_n]}T^{2-2^n}$. Besides, $\Psi_{V_l,0,2} = \tau_{[.w_nl]}$. Hence

$$\begin{aligned}\tau_{[.w_{n+1}l]} &= T^{1-2^n}\Psi_{U\cap V_l,-1,1}\,T^{2^n-1} = T^{1-2^n}\Psi_{V_l,0,2}\Psi^{-1}_{U,-2,0}\Psi^{-1}_{V_l,0,2}\Psi_{U,-2,0}\,T^{2^n-1}\\ &= T^{1-2^n}\tau_{[.w_nl]}(T^{2^n-2}\tau_{[.w_nl_n]}T^{2-2^n})^{-1}\tau^{-1}_{[.w_nl]}(T^{2^n-2}\tau_{[.w_nl_n]}T^{2-2^n})T^{2^n-1}\\ &= T^{1-2^n}\tau_{[.w_nl]}T^{2^n-2}\tau^{-1}_{[.w_nl_n]}T^{2-2^n}\tau^{-1}_{[.w_nl]}T^{2^n-2}\tau_{[.w_nl_n]}T,\end{aligned}$$

which is in the group $H_n$.

Next we are going to show that $\tau_{[.w_{n-1}l]} \in H_n$ for each $l \in \{b,c,d\}$ (so that $H_{n-1} \subset H_n$). Note that exactly one of the letters $l_{n-2}$, $l_{n-1}$, and $l_n$ coincides with $l$. In view of Lemmas 2.11, 2.12, and 2.13, the cylinder $[.w_{n-1}l]$ is a disjoint union of one (if $l = l_n$), two (if $l = l_{n-2}$), or three (if $l = l_{n-1}$) sets of the form $T^N([.w_nl'])$, where $l' \in \{b,c,d\}$ and $N \in \mathbb{Z}$. By Lemma 3.1, $\tau_{T^N([.w_nl'])} = T^N\tau_{[.w_nl']}T^{-N}$, which belongs to $H_n$. Then it follows from Lemma 3.2 that $\tau_{[.w_{n-1}l]}$ is a product of at most three elements of the group $H_n$. Hence $\tau_{[.w_{n-1}l]} \in H_n$ as well.

We have shown that $H_{n+1} \subset H_n$ and $H_{n-1} \subset H_n$ for all $n \geq 4$. As a consequence, $H_{n+1} = H_n$ for $n \geq 4$. It follows by induction that $H_n = H_4$ for $n \geq 4$. ■

**Lemma 4.5** *$G_n = G_3$ for all $n > 3$.*

**Proof.** First we are going to show that $\tau_{[.w_4l]} \in G_3$ for each $l \in \{b,c,d\}$ (so that the group $H_4$ is a subgroup of $G_3$). Let $U = [acabacad.]$ and $V_l = [.acabacal]$. Since $w_3 = acabaca$ and $l_3 = d$, we have $U = T^8([.w_3l_3])$, $V_l = [.w_3l]$, and $U \cap V_l = [w_3l_3.w_3l] = T^8([.w_3l_3w_3l]) = T^8([.w_4l])$. By Lemma 2.5, any occurrence of $w_3l_3$ in $\xi$ is immediately followed by $w_3b$, $w_3c$, or $w_3d$. As a consequence, $U = [w_3l_3.]$ is contained in $[.w_3]$. Clearly, $V_l \subset [.w_3]$ as well. Observe that the cylinder $[.w_3] = [.acabaca]$ is disjoint from $T([.w_3]) = [a.cabaca]$ and $T^2([.w_3]) = [ac.abaca]$. Since $U$ and $V_l$ are subsets of $[.w_3]$, the cylinder $U$ is disjoint from $T(V_l)$ and $T^2(V_l)$. Then it follows from Lemma 3.4 that

$$\Psi_{V_l,0,1}\Psi^{-1}_{U,-1,0}\Psi^{-1}_{V_l,0,1}\Psi_{U,-1,0} = \Psi_{U\cap V_l,-1,1}.$$

By Lemma 3.1, $\Psi_{U\cap V_l,-1,1} = T^7\tau_{[.w_4l]}T^{-7}$ and $\Psi_{U,-1,0} = T^7\delta_{[.w_3l_3]}T^{-7}$. Besides, $\Psi_{V_l,0,1} = \delta_{[.w_3l]}$. Hence

$$\begin{aligned}\tau_{[.w_4l]} &= T^{-7}\Psi_{U\cap V_l,-1,1}\,T^7 = T^{-7}\Psi_{V_l,0,1}\Psi^{-1}_{U,-1,0}\Psi^{-1}_{V_l,0,1}\Psi_{U,-1,0}\,T^7\\ &= T^{-7}\delta_{[.w_3l]}(T^7\delta_{[.w_3l_3]}T^{-7})^{-1}\delta^{-1}_{[.w_3l]}(T^7\delta_{[.w_3l_3]}T^{-7})T^7\\ &= T^{-7}\delta_{[.w_3l]}T^7\delta_{[.w_3l_3]}T^{-7}\delta_{[.w_3l]}T^7\delta_{[.w_3l_3]},\end{aligned}$$

which is in the group $G_3$.

Next we derive three formulas. By Lemma 2.11, $[.w_nl_n] = [.w_{n+1}b] \sqcup [.w_{n+1}c] \sqcup [.w_{n+1}d]$ for all $n \geq 1$. Then Lemma 3.2 implies that $\delta_{[.w_nl_n]} = \delta_{[.w_{n+1}b]}\delta_{[.w_{n+1}c]}\delta_{[.w_{n+1}d]}$ for $n \geq 1$. By Lemma 2.12, $[.w_nl_{n+1}] = T^{2^n}([.w_{n+1}l_{n+1}])$ for all $n \geq 1$. Then Lemma 3.1 implies that $\delta_{[.w_nl_{n+1}]} = T^{2^n}\delta_{[.w_{n+1}l_{n+1}]}T^{-2^n}$ for $n \geq 1$. By Lemma 2.13, $[.w_nl_{n-1}] = T^{2^n}([.w_{n+1}l_{n-1}]) \sqcup T^{3\cdot 2^{n-1}}([.w_{n+1}l_{n-1}])$ for all $n \geq 2$. Then Lemmas 3.1 and 3.2 imply that

$$\begin{aligned}\delta_{[.w_nl_{n-1}]} &= (T^{2^n}\delta_{[.w_{n+1}l_{n-1}]}T^{-2^n})(T^{3\cdot 2^{n-1}}\delta_{[.w_{n+1}l_{n-1}]}T^{-3\cdot 2^{n-1}})\\ &= T^{2^n}\delta_{[.w_{n+1}l_{n-1}]}T^{2^{n-1}}\delta_{[.w_{n+1}l_{n-1}]}T^{-3\cdot 2^{n-1}}\end{aligned}$$

for $n \geq 2$. Note that for any $n \geq 2$ the triple $l_{n-1}, l_n, l_{n+1}$ is a permutation of the triple $b, c, d$. Therefore the above three formulas imply that transformations $\delta_{[.w_n b]}$, $\delta_{[.w_n c]}$, and $\delta_{[.w_n d]}$ belong to the group $G_{n+1}$. Hence $G_n \subset G_{n+1}$ for all $n \geq 2$.

Next we are going to show that $G_{n+1} \subset G_n$ for all $n \geq 3$. By the above, $H_4 \subset G_3$. In view of Lemmas 4.3 and 4.4, the group $G_3$ contains $\delta_{[.w_{n+1} l_n]}$ for all $n \geq 2$. Since $G_n \subset G_{n+1}$ for $n \geq 2$, it follows by induction that $G_3 \subset G_n$ for all $n \geq 3$. As a consequence, $\delta_{[.w_{n+1} l_n]} \in G_n$ for $n \geq 3$. Besides, for any $n \geq 1$ we have $\delta_{[.w_{n+1} l_{n+1}]} = T^{-2^n} \delta_{[.w_n l_{n+1}]} T^{2^n}$, which belongs to $G_n$. Therefore for any $n \geq 3$ the group $G_n$ contains two of the three transformations $\delta_{[.w_{n+1} b]}$, $\delta_{[.w_{n+1} c]}$, and $\delta_{[.w_{n+1} d]}$. Since the product of all three is $\delta_{[.w_{n+1} b]} \delta_{[.w_{n+1} c]} \delta_{[.w_{n+1} d]} = \delta_{[.w_n l_n]} \in G_n$, the remaining one of the three is in $G_n$ as well. Hence $G_n$ contains all generators of the group $G_{n+1}$ so that $G_{n+1} \subset G_n$.

We have shown that $G_n \subset G_{n+1}$ for $n \geq 2$ and $G_{n+1} \subset G_n$ for $n \geq 3$. As a consequence, $G_{n+1} = G_n$ for $n \geq 3$. It follows by induction that $G_n = G_3$ for all $n \geq 3$. ∎

**Proof of Theorem 1.10.** According to Theorem 1.4, the topological full group $[[T]]$ is generated by $T$ and all transformations of the form $\delta_U$, where $U \subset \Omega$ is a clopen set. By Lemma 4.2, each $\delta_U$ is contained in the group $G_n$ (generated by $\delta_{[.w_n b]}$, $\delta_{[.w_n c]}$, $\delta_{[.w_n d]}$, and $T$) for $n$ large enough. Then it follows from Lemma 4.5 that each $\delta_U$ is contained in the group $G_3$. We conclude that $G_3 = [[T]]$. By Lemma 4.1, the group $G_3$ coincides with the group generated by $T$, $\delta_{[.b]}$, $\delta_{[.d]}$, and $\delta_{[.acacac]}$. ∎

# References


[GPS] T. Giordano, I. F. Putnam, and C. F. Skau, Full groups of Cantor minimal systems. *Israel J. Math.* **111** (1999), no. 1, 285–320.

[Lys] I. G. Lysenok, A system of defining relations for a Grigorchuk group. *Math. Notes* **38** (1985), 784–792 [translated from *Mat. Zametki* **38** (1985), no. 4, 503–516].

[M-B] N. Matte Bon, Topological full groups of minimal subshifts with subgroups of intermediate growth. *J. Modern Dynamics* **9** (2015), 67–80.

[Mat] H. Matui, Some remarks on topological full groups of Cantor minimal systems. *Internat. J. Math.* **17** (2006), no. 2, 231–251.

[Vor] Ya. Vorobets, On a substitution subshift related to the Grigorchuk group. *Proc. Steklov Inst. Math.* **271** (2010), 306–321.



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