On stability of periodic billiard orbits in polyhedra

Suppose a group G is generated by elements $a_1, \dots, a_n$. Then any element $g \neq 1$ of G is represented as a product $g_1 g_2 \dots g_k$, where each g_i is a generator a_j or an inverse a_j^{-1}. The smallest k that allows such a representation is called the *length* of g. The length of the unit element is set to 0. Notice that the length depends on the set of generators.

The group G generated by $a_1, \dots, a_n$ is called a *free group* with n generators ($a_1, \dots, a_n$ are called *free generators*) if for any group H and any $h_1, \dots, h_n \in H$ there exists a unique homomorphism $f : G \to H$ such that $f(a_i) = h_i$, $1 \leq i \leq n$. A nontrivial element $g \in G$ is represented as $a_{i_1}^{m_1} a_{i_2}^{m_2} \dots a_{i_l}^{m_l}$, where $l \geq 1$, $1 \leq i_j \leq n$ and $m_j \neq 0$ for $1 \leq j \leq l$, and $i_j \neq i_{j+1}$ for $1 \leq j < l$. The group G is free if and only if such a representation is unique for any $g \neq 1$.

For any $\phi \in [0, 2\pi)$ let

$$A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

A_ϕ and B_ϕ are matrices of two rotations in $\mathbb{R}^3$ by the angle ϕ.

Proposition 1 *For all but countably many angles $\phi \in [0, 2\pi)$, the subgroup of $SO(3)$ generated by A_ϕ and B_ϕ is a free group with two generators.*

Proof. Let G be a free group with two generators a and b. For any $\phi \in [0, 2\pi)$ let $F_\phi : G \to SO(3)$ be the homomorphism such that $F_\phi(a) = A_\phi$, $F_\phi(b) = B_\phi$. We have to prove that F_ϕ is injective for all but countably many angles ϕ. For any $g \in G$ let $F_\phi(g) = (c_{jk}[g](\phi))_{1 \leq j,k \leq 3}$. Then $c_{jk}[g]$ are functions on $[0, 2\pi)$.

Let $n \geq 1$ be an integer. Recall that a trigonometric polynomial of degree n is a function $p : \mathbb{R} \to \mathbb{C}$ such that $p(\phi) = \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos k\phi + \beta_k \sin k\phi)$ for some $\alpha_k, \beta_k \in \mathbb{C}$, where $(\alpha_n, \beta_n) \neq (0, 0)$. A trigonometric polynomial of degree 0 is a constant function. The degree of a trigonometric polynomial p is denoted by $\deg p$. Since $\cos k\phi = (e^{ik\phi} + e^{-ik\phi})/2$, $\sin k\phi = (e^{ik\phi} - e^{-ik\phi})/(2i)$, a function p is a trigonometric polynomial of degree $n \geq 1$ if and only if $p(\phi) = \sum_{k=-n}^{n} \alpha_k e^{ik\phi}$ for some $\alpha_k \in \mathbb{C}$, where $(\alpha_n, \alpha_{-n}) \neq (0, 0)$. Hence if p and q are trigonometric polynomials, then so are $p + q$ and pq. Moreover, $\deg(p + q) \leq \max(\deg p, \deg q)$, $\deg pq \leq \deg p + \deg q$. If $\deg q < \deg p$ then $\deg(p + q) = \deg p$. It is possible that $\deg pq < \deg p + \deg q$, for example,

$(\cos\phi + i\sin\phi)(\cos\phi - i\sin\phi) = 1$. However for any integers $m_1, \dots, m_l > 0$ the product $\cos m_1\phi \cos m_2\phi \dots \cos m_l\phi$ is a trigonometric polynomial in ϕ of degree $m_1 + \cdots + m_l$. Indeed, the equality $2\cos m\phi \cos m'\phi = \cos(m+m')\phi + \cos(m-m')\phi$ implies $\cos m_1\phi \cos m_2\phi \dots \cos m_l\phi - 2^{1-l}\cos(m_1 + \cdots + m_l)\phi$ is a trigonometric polynomial in ϕ of degree less than $m_1 + \cdots + m_l$.

We claim that for any $g \in G$ the function $c_{22}[g]$ is a trigonometric polynomial of degree $|g|$, where $|g|$ is the length of g. It is easy to see that $c_{22}[a^n](\phi) = c_{22}[b^n](\phi) = \cos n\phi$ for all $n \in \mathbb{Z}$. Now suppose $g = g_1^{m_1} \dots g_l^{m_l}$, where $l > 1$, $m_j \neq 0$ for $1 \leq j \leq l$, $\{g_1, g_2\} = \{a, b\}$, $g_j = g_1$ if j is odd and $g_j = g_2$ if j is even. Then

$$c_{22}[g] = \sum_{1 \leq j_1, \dots, j_{l-1} \leq 3} c_{2j_1}[g_1^{m_1}] c_{j_1 j_2}[g_2^{m_2}] \dots c_{j_{l-1}2}[g_l^{m_l}].$$

It is easy to see that $c_{ks}[g_j^{m_j}]$ is a trigonometric polynomial of degree at most $|m_j|$. Therefore each summand in the above sum is a trigonometric polynomial of degree at most $|m_1| + \cdots + |m_l| = |g|$. Let $p = c_{22}[g_1^{m_1}] \dots c_{22}[g_l^{m_l}]$. Clearly, $p(\phi) = \cos m_1\phi \dots \cos m_l\phi$, hence $\deg p = |g|$. Consider a set of indices $j_1, \dots, j_{l-1} \in \{1, 2, 3\}$ such that $j_k \neq 2$ for some k. It is no loss to assume that $j_s = 2$ for $1 \leq s < k$. In the cases $g_k = a$, $j_k = 3$ and $g_k = b$, $j_k = 1$, we have $c_{2j_k}[g_k^{m_k}] = 0$. In the cases $g_k = a$, $j_k = 1$ and $g_k = b$, $j_k = 3$, we have $c_{j_k j_{k+1}}[g_{k+1}^{m_{k+1}}] = 0$ or 1 (here $j_{k+1} = 2$ if $k + 1 = l$). In any case $c_{2j_1}[g_1^{m_1}] c_{j_1 j_2}[g_2^{m_2}] \dots c_{j_{l-1}2}[g_l^{m_l}]$ is a trigonometric polynomial of degree less than $|g|$. It follows that $\deg(c_{22}[g] - p) < |g|$, hence $\deg c_{22}[g] = |g|$.

Let $g \in G$, $g \neq 1$. Then $|g| \geq 1$. Since $c_{22}[g]$ is a trigonometric polynomial of degree $n = |g|$, so is $c_{22}[g] - 1$. It follows that $c_{22}[g](\phi) - 1 = e^{-in\phi}P(e^{i\phi})$, where P is a nonzero polynomial of degree at most $2n$. Therefore $c_{22}[g](\phi) = 1$ for at most $2n$ values of $\phi \in [0, 2\pi)$. Clearly, $F_\phi(g) = 1$ only if $c_{22}[g](\phi) = 1$. Hence for all but countably many angles $\phi \in [0, 2\pi)$ we have $F_\phi(g) = 1$ only if $g = 1$. The latter property implies F_ϕ is injective. ∎

Lemma 2 *Suppose G is a free group with two generators a and b. Then the subgroup of G generated by elements $g_k = b^k a b^{-k}$, $1 \leq k \leq n$, is a free group with n generators.*

Proof. Suppose $g = g_{i_1}^{m_1} g_{i_2}^{m_2} \dots g_{i_l}^{m_l}$, where $l \geq 1$, $1 \leq i_j \leq n$ and $m_j \neq 0$ for $1 \leq j \leq l$, and $i_j \neq i_{j+1}$ for $1 \leq j < l$. We have to prove that $g \neq 1$. If $l = 1$ then $g = b^{i_1} a^{m_1} b^{-i_1}$. Otherwise $g = b^{i_1} a^{m_1} b^{i_2 - i_1} a^{m_2} \dots b^{i_l - i_{l-1}} a^{m_l} b^{-i_l}$. Since none of the integers $i_1, i_2 - i_1, \dots, i_l - i_{l-1}, -i_l$ and $m_1, \dots, m_l$ is equal to zero, it follows that $g \neq 1$. ∎

Suppose a group G is generated by n elements $g_1, \dots, g_n$ of order 2. The group G is called the *free product* of n groups of order 2 (we say that $g_1, \dots, g_n$ *freely generate* G) if $G = \langle g_1, \dots, g_n \mid g_1^2 = \dots = g_n^2 = 1 \rangle$ or, equivalently, for any group H and any elements $h_1, \dots, h_n \in H$ of order 2 there exists a unique homomorphism $f : G \to H$ such that $f(g_i) = h_i$, $1 \le i \le n$. A nontrivial element $g \in G$ is represented as $g_{i_1} g_{i_2} \dots g_{i_l}$, where $l \ge 1$, $1 \le i_j \le n$ for $1 \le j \le l$, and $i_j \ne i_{j+1}$ for $1 \le j < l$. The group G is freely generated by n involutions if and only if such a representation is unique for any $g \ne 1$.

Lemma 3 *Suppose G is a group generated by $n \ge 2$ elements $g_1, \dots, g_n$ of order 2. Then G is freely generated by n involutions if and only if elements $g_1 g_2, g_1 g_3, \dots, g_1 g_n$ generate a free group with $n - 1$ generators.*

Proof. Let H denote the subgroup of G generated by elements $h_i = g_1 g_i$, $2 \le i \le n$. Consider an element $h = h_{i_1}^{\varepsilon_1} h_{i_2}^{\varepsilon_2} \dots h_{i_l}^{\varepsilon_l}$, where $l \ge 1$, $2 \le i_j \le n$, $\varepsilon_j = 1$ or -1, and $\varepsilon_j = \varepsilon_{j+1}$ whenever $i_j = i_{j+1}$. Since $h_i = g_1 g_i$ and $h_i^{-1} = g_i g_1$ for $2 \le i \le n$, we have $h = g'_0 g_{i_1} g'_1 \dots g_{i_l} g'_l$, where $g'_j = g_1$ or 1, $0 \le j \le l$. Moreover, $g'_j = g_1$ whenever $\varepsilon_j = \varepsilon_{j+1}$. In particular, $h \ne 1$ if G is freely generated by the involutions $g_1, \dots, g_n$. It follows that H is a free group with $n - 1$ generators if G is freely generated by $g_1, \dots, g_n$.

Now suppose H is the free group with free generators $h_2, \dots, h_n$. To prove that G is freely generated by n involutions, it is sufficient to show that $g \ne 1$ whenever $g = g_{i_1} \dots g_{i_l}$, where $l \ge 1$, $1 \le i_j \le n$, $i_j \ne i_{j+1}$. Note that $g_i g_j = h_i^{-1} h_j$ for $1 \le i, j \le n$, where by definition $h_1 = 1$. On the other hand, none of the elements $g_1, \dots, g_n$ belongs to H as a free group has no elements of order 2. Hence if l is odd then $g \notin H$, in particular, $g \ne 1$. Consider the case when l is even. Here $g = h_{i_1}^{-1} h_{i_2} \dots h_{i_{l-1}}^{-1} h_{i_l} \in H$. It is easy to see that the length of g in H is equal to the number of indices $j \in \{1, \dots, l\}$ such that $i_j \ne 1$. As this number is positive, $g \ne 1$. ■

A vector plane in $\mathbb{R}^3$ is uniquely determined by the orthogonal straight line. Therefore the set of planes in $\mathbb{R}^3$ is parametrized by the projective plane $\mathbb{R}P^2$. Recall that elements of $\mathbb{R}P^2$ are one-dimensional subspaces of $\mathbb{R}^3$, i.e., straight lines passing through the origin. For any straight line $\gamma \in \mathbb{R}P^2$ let $R[\gamma] \in O(3)$ be the matrix of the reflection of $\mathbb{R}^3$ in the vector plane orthogonal to γ. Given $\gamma_1, \dots, \gamma_n \in \mathbb{R}P^2$, let $J[\gamma_1, \dots, \gamma_n]$ denote the homomorphism of the group $\mathbb{Z}_2^{*n} = \langle a_1, \dots, a_n \mid a_1^2 = \dots = a_n^2 = 1 \rangle$ to $O(3)$ such that $J[\gamma_1, \dots, \gamma_n](a_i) = R[\gamma_i]$, $1 \le i \le n$.

Theorem 4 *There exists a dense G_δ-set $U_n \subset (\mathbb{R}P^2)^n$ such that for any $(\gamma_1, \dots, \gamma_n) \in U_n$ the homomorphism $J[\gamma_1, \dots, \gamma_n]$ is injective.*

Proof. The case $n = 1$ is trivial as $J[\gamma]$ is always injective. Consider the case $n > 1$. Let γ_1 denote the x-axis in $\mathbb{R}^3$. For any $\phi \in [0, 2\pi)$ the matrices $B_\phi A_\phi B_\phi^{-1}, B_\phi^2 A_\phi B_\phi^{-2}, \dots, B_\phi^{n-1} A_\phi B_\phi^{-(n-1)} \in SO(3)$ are matrices of rotations by ϕ about axes orthogonal to γ_1. It follows that there exist straight lines $\gamma_2, \dots, \gamma_n \in \mathbb{R}P^2$ such that $B_\phi^{k-1} A_\phi B_\phi^{-(k-1)} = R[\gamma_1]R[\gamma_k]$ for $2 \le k \le n$. By Proposition 1, we can choose ϕ so that A_ϕ and B_ϕ generate a free subgroup of $SO(3)$ with two generators. Then Lemmas 2 and 3 imply that the subgroup of $O(3)$ generated by $R[\gamma_1], \dots, R[\gamma_n]$ is freely generated by n involutions. This means $J[\gamma_1, \dots, \gamma_n]$ is an injective map.

Given $g \in \mathbb{Z}_2^{*k}$, let $U_n(g)$ denote the set of $(\gamma_1, \dots, \gamma_n) \in (\mathbb{R}P^2)^n$ such that $J[\gamma_1, \dots, \gamma_n](g) \neq 1$. Notice that $\mathbb{R}P^2$ is an analytic manifold. For any $\gamma \in \mathbb{R}P^2$ the orthogonal reflection of $\mathbb{R}^3$ in the vector plane orthogonal to γ is given by the formula $v \mapsto v - 2\frac{(v, n_\gamma)}{(n_\gamma, n_\gamma)} n_\gamma$, where $(\cdot, \cdot)$ denotes the scalar product and n_γ is a nonzero vector parallel to γ. The formula shows that $R[\gamma]$ is a real analytic matrix-valued function of $\gamma \in \mathbb{R}P^2$. Therefore $J[\gamma_1, \dots, \gamma_n](g)$ is a real analytic matrix-valued function of $(\gamma_1, \dots, \gamma_n) \in (\mathbb{R}P^2)^n$. Since $\mathbb{R}P^2$ is connected, it follows that $U_n(g)$ is an open set that is either dense in $(\mathbb{R}P^2)^n$ or empty. By the above $J[\gamma_1, \dots, \gamma_n]$ is injective for some $\gamma_1, \dots, \gamma_n \in \mathbb{R}P^2$. This implies $U_n(g)$ is not empty for any $g \neq 1$. Hence $U_n = \bigcap_{g \neq 1} U_n(g)$ is a dense G_δ-set. Obviously, $J[\gamma_1, \dots, \gamma_n]$ is injective for any $(\gamma_1, \dots, \gamma_n) \in U_n$. ∎

Let Π be a polyhedron in $\mathbb{R}^3$. A *billiard orbit* in Π is a (finite or infinite) broken line $x_0 x_1 \dots$ such that (i) each x_i lies on the boundary of Π, moreover, x_i is an interior point of a face unless it is an endpoint of the broken line, (ii) the interior of each segment $x_i x_{i+1}$ is contained in the interior of Π, and (iii) the unit vectors $\frac{x_{i+1} - x_i}{|x_{i+1} - x_i|}$ and $\frac{x_{i+2} - x_{i+1}}{|x_{i+2} - x_{i+1}|}$ are symmetric with respect to the orthogonal reflection in the face containing x_{i+1}. The latter condition (the reflection rule) means that a billiard orbit is a trajectory of a point-mass moving freely within Π subject to elastic rebounds off the faces. One or both endpoints of a billiard orbit in Π may lie on edges of Π. A billiard orbit is called *singular* if this is the case. Any finite nonsingular billiard orbit can be continued to a singular or infinite one. A finite billiard orbit $x_0 x_1 \dots x_n$ is called *periodic* if $x_n = x_0$ and the infinite periodic broken line $x_0 x_1 \dots x_n x_1 \dots x_n x_1 \dots$ is a billiard orbit.

Suppose that faces of the polyhedron are labelled by elements of a finite set $\mathcal{A}$. Then each nonsingular billiard orbit $x_0 x_1 x_2 \dots$ is assigned a *code word* $w = a_1 a_2 \dots$, where $a_i \in \mathcal{A}$ labels the face containing x_i.

Suppose Π_1 and Π_2 are polyhedra in $\mathbb{R}^3$. Given $\varepsilon > 0$, we say that a face

f_1 of Π_1 is ε-*close* to a face f_2 of Π_2 if the distance from any point of f_1 to f_2 is less than ε and the distance from any point of f_2 to f_1 is less than ε. Note that ε-close faces need not have the same number of edges. Further, we say that the polyhedron Π_2 is an ε-*perturbation* of Π_1 if Π_2 have the same number of faces as Π_1, and each face of Π_2 is ε-close to a face of Π_1. Assume ε is so small that distinct faces of Π_1 are not 2ε-close. Then ε-closeness provides a one-to-one correspondence between faces of Π_1 and Π_2. In particular, any labeling of faces of Π_1 induces a labeling of faces of Π_2. We shall say that a property of polyhedra can be achieved by an *arbitrarily small perturbation of faces* of a polyhedron Π if for any $\varepsilon > 0$ there exists an ε-perturbation of Π with this property.

Lemma 5 *Let Π be a polyhedron in $\mathbb{R}^3$ and $f_1, \dots, f_m$ be faces of Π. Let $\gamma_1, \dots, \gamma_m \in \mathbb{R}P^2$ be straight lines such that γ_i is orthogonal to f_i. Then for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property: given $\tilde{\gamma}_1, \dots, \tilde{\gamma}_m \in \mathbb{R}P^2$ such that the angle between γ_i and $\tilde{\gamma}_i$ is less than δ, there exists an ε-perturbation $\widetilde{\Pi}$ of Π such that each $\tilde{\gamma}_i$ is orthogonal to a face of $\widetilde{\Pi}$.*

Let Π be a polyhedron and $x_0 x_1 \dots x_n$ be a finite nonsingular billiard orbit in Π. Given $\delta > 0$, there exists $\varepsilon > 0$ such that any ε-perturbation of Π admits a nonsingular billiard orbit $y_0 y_1 \dots y_n$ such that y_i and x_i lie on ε-close faces and $|y_i - x_i| < \delta$. Assuming that faces of Π are labelled and ε is small enough, $y_0 y_1 \dots y_n$ has the same code word as $x_0 x_1 \dots x_n$. In the case $x_0 x_1 \dots x_n$ is a periodic billiard orbit, we wish to know whether the billiard orbit $y_0 y_1 \dots y_n$ can be chosen periodic too. The following theorem gives a partial answer to this question.

Theorem 6 *Suppose there is a periodic billiard orbit with a code word w in a polyhedron Π. Then there exists an arbitrarily small perturbation of faces of Π such that the perturbed polyhedron admits at most one periodic billiard orbit with the code word w if w has even length, and no periodic billiard orbit with the code word w if w has odd length.*

Proof. Let $\mathcal{A}$ be the set whose elements label faces of Π. For any $a \in \mathcal{A}$ let f_a denote the face with label a and S_a denote the orthogonal reflection of $\mathbb{R}^3$ in the affine plane containing f_a. Let $w = a_1 \dots a_n$. Obviously, $a_i \neq a_{i+1}$ for $1 \leq i < n$, and $a_n \neq a_1$. Suppose $x_0 x_1 \dots x_n$ is a nonsingular billiard orbit in Π such that $x_0 \in f_{a_n}$ and $x_i \in f_{a_i}$, $1 \leq i \leq n$. We apply the so-called unfolding procedure to this orbit. Namely, consider polyhedra $\Pi_0 = \Pi, \Pi_1, \dots, \Pi_n$ and affine orthogonal transformations $T_0 = \mathrm{id}, T_1, \dots, T_n$ such that $\Pi_i = T_i(\Pi)$ $(1 \le i \le n)$ and Π_i is symmetric to Π_{i-1} with respect to their common face $T_{i-1}(f_{a_i})$. The polyhedra and the transformations are uniquely determined. By construction, $T_i T_{i-1}^{-1} = T_{i-1} S_{a_i} T_{i-1}^{-1}$. Hence $T_i = S_{a_1} S_{a_2} \dots S_{a_i}$, $1 \le i \le n$. Now let $x_i' = T_i(x_i)$, $1 \le i \le n$. Then $x_0 x_1' = x_0 x_1 \subset \Pi$ and $x_i' x_{i+1}' = T_i(x_i x_{i+1}) \subset \Pi_i$ for $1 \le i < n$. The reflection rule implies that segments $x_0x_1', x_1'x_2', \dots, x_{n-1}'x_n'$ form the single segment $x_0 x_n'$, which is called the unfolding of the billiard orbit $x_0 x_1 \dots x_n$. Now suppose $x_n = x_0$. Then the vector $T_n x_0 - x_0$ is of the same direction as $x_1 - x_0$. It is easy to see that $x_0 x_1 \dots x_n$ is a periodic billiard orbit if and only if the segment $T_n(x_0 x_1) = x_n' T_n(x_1) \subset \Pi_n$ continues $x_0 x_n'$. An equivalent condition is that $T_n(x_1) - T_n(x_0) = x_1 - x_0$, i.e., the vector $x_1 - x_0$ is invariant under the linear part L of T_n. In particular, 1 has to be an eigenvalue of L.

Suppose $x_0 x_1 \dots x_n$ and $y_0 y_1 \dots y_n$ are distinct periodic billiard orbits in Π with the code word w. Their unfoldings are $x_0 T_n(x_0)$ and $y_0 T_n(y_0)$. Since the two orbits are distinct, it follows that $y_0 \ne x_0$. By the above vectors $T_n(x_0) - x_0$ and $T_n(y_0) - y_0$ are invariant under L. So is the vector $Lv - v$, where $v = y_0 - x_0$. Hence $(L-1)^2 v = 0$. Since L is an orthogonal operator, we obtain $Lv = v$. Note that v is parallel to the face f_{a_n} while $T_n(x_0) - x_0$ is not. This implies that 1 is a multiple eigenvalue of L.

For any $a \in \mathcal{A}$ let $\gamma_a \in \mathbb{R}P^2$ be the straight line orthogonal to the plane containing the face f_a. Then $R[\gamma_a]$ is the matrix of the linear part of S_a. Consequently, $M = R[\gamma_{a_1}] R[\gamma_{a_2}] \dots R[\gamma_{a_n}]$ is the matrix of the linear part of T_n. Assume that the involutive matrices $R[\gamma_a]$, $a \in \mathcal{A}$, freely generate a subgroup of $O(3)$. Since $a_i \ne a_{i+1}$ and $a_n \ne a_1$, it follows that $M \ne 1$ and $M^2 \ne 1$. Hence if n is even then 1 is a simple eigenvalue of the matrix M. If n is odd then 1 is not an eigenvalue of M.

By Theorem 4 and Lemma 5, there exists an arbitrarily small perturbation of faces of Π such that matrices of the linear parts of orthogonal reflections in faces of the perturbed polyhedron $\widetilde{\Pi}$ freely generate a subgroup of $O(3)$. It follows from the above that the polyhedron $\widetilde{\Pi}$ admits at most one periodic billiard orbit with any fixed code word of even length and no periodic billiard orbits with code words of odd length. ∎