Automata generating free products of groups of order 2

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Abstract

We construct a family of automata with n states, $n \geq 4$, acting on a rooted binary tree that generate the free products of cyclic groups of order 2.

Introduction

An automaton group is the group generated by transformations defined by all states of a finite invertible automaton (see precise definitions in Section 1) over a finite alphabet X. The transformations act on the set X^* of finite words over X, which can be regarded as a regular rooted tree. The first mentioning of automaton groups dates back to early 1960s [Glu61, Hoř63]. Until the beginning of 1980s the interest in these groups was somewhat sporadic. It started to grow rapidly after it was shown that the class of automaton groups contains counterexamples to the general Burnside problem [Ale72, Sus79, Gri80, GS83]. Later, Grigorchuk in [Gri84] solved Milnor problem on the existence of groups of intermediate growth by producing series of groups generated by automata. A very abridged list of automaton groups that have extraordinary properties includes: amenable, but not elementary amenable group (see [Gri98]) answering Day problem [Day57]; amenable, but not subexponentially amenable

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Basilica group [GŻ02, BV05]; lamplighter group that gave rise to the negative solution of the strong Atiyah conjecture on L_2-Betti numbers [GLSŻ00].

All transformations defined by states of finite invertible automata over a fixed alphabet form a group of automatic transformations over this alphabet. The structure of this large group is yet to be understood. An interesting question is the embedding of known groups into this group. For example, Brunner and Sidki proved in [BS98] that $GL_n(\mathbb{Z})$ can be generated by finite automata over the alphabet with 2^n letters. In this paper we address this question in regard to the free products of groups of order 2. The first embedding of such free products into the group of automatic transformations over the 2-letter alphabet was constructed by Olijnyk [Olī99]. We also mention results of C. Gupta, N. Gupta and A. Olijnyk [Oli00, GGO07] who embedded the free product of any finite family of finite groups into a group of automatic transformations over a suitable alphabet.

The above constructions lack the important property of self-similarity [Nek05]. In other words, the group is not generated by all states of a single automaton. The first self-similar example was provided by a 3-state automaton $\mathcal{B}_3$ over 2-letter alphabet whose Moore diagram is depicted in Figure 1. This automaton was studied during the summer school in Automata groups held in 2004 at the Autonomous University of Barcelona in Bellaterra. Since then, it is known as the Bellaterra automaton. It was proved by Muntyan (see the proof in [BGK$^+$07] or [Nek05]) that $\mathcal{B}_3$ generates the group isomorphic to the free product of 3 copies of groups of order 2.

Many papers on free groups and free products generated by automata share the same idea of dual automaton. For an automaton $\mathcal{A}$ the dual automaton $\hat{\mathcal{A}}$ is obtained from $\mathcal{A}$ by interchanging the states and the alphabet, and swapping the transition and output functions. For precise definition see Section 1. It turns out that the "freeness" properties of the group generated by $\mathcal{A}$ are related to certain transitivity conditions of the action of the group generated by $\hat{\mathcal{A}}$.

The Bellaterra automaton belongs to the class of bireversible automata [Nek05, GM05], which seems to be a natural source for automata generating free groups and free products. An invertible automaton is called bireversible if its dual and the dual to its inverse are also invertible. It is worth mentioning that the Bellaterra automaton was discovered while classifying all bireversible 3-state automata over 2-letter alphabet.

The transformations a, b, c defined by states of the Bellaterra automaton act on the set $\{0,1\}^*$ of finite words over the alphabet $X = \{0,1\}$. They are uniquely determined by so-called wreath recursion

$$\begin{array}{rcl} a & = & (c,b), \\ b & = & (b,c), \\ c & = & (a,a)\sigma, \end{array}$$

where $t = (t_0, t_1)$ means that $t(0w) = 0t_0(w)$ and $t(1w) = 1t_1(w)$ for any word $w \in X^*$ while $t = (t_0, t_1)\sigma$ means $t(0w) = 1t_0(w)$ and $t(1w) = 0t_1(w)$ for any $w \in X^*$.

The Bellaterra automaton gives rise to a family of bireversible automata in which all states define involutive transformations. The construction is very simple. Namely, we modify the automaton $\mathcal{B}_3$ by inserting new states on the path from c to a. More precisely, each automaton in the family is defined by wreath recursion

$$\begin{array}{lcl} a & = & (c,b), \\ b & = & (b,c), \\ c & = & (q_1,q_1)\sigma_0, \\ q_i & = & (q_{i+1},q_{i+1})\sigma_i, \;\; i = 1,\ldots,n-4, \\ q_{n-3} & = & (a,a)\sigma_{n-3}, \end{array} \tag{1}$$

where $\sigma_i \in \mathrm{Sym}(\{0,1\})$ is chosen arbitrarily.

Conjecturally, each automaton in the family for which at least one of the σ_i is nontrivial, generates the free product of groups of order 2. The first result supporting this conjecture was obtained by M. Vorobets and Y. Vorobets [VV06]. It was shown that if the number of states is odd and $\sigma_i = (12)$ for all i, then the conjecture holds. In the subsequent paper by the same authors and B. Steinberg [SVV06] the conjecture was proved for the automata with even number of states and additional condition that the number of nontrivial σ_i is odd.

In this paper we prove that any n-state automaton from the family (1) with $n \geq 4$ satisfying $\sigma_0 = (12)$ and $\sigma_{n-3} = (12)$ generates the free product of groups of order 2. This result covers the series constructed in [VV06] except one, but the most important case $n = 3$, and partially overlaps with a family constructed in [SVV06]. More precisely, our main results are as follows.

Let $\mathcal{B}_4$ be the automaton defined by the wreath recursion

$$\begin{array}{rcl} a & = & (c, b), \\ b & = & (b, c), \\ c & = & (d, d)\sigma, \\ d & = & (a, a)\sigma. \end{array}$$

Theorem 0.1. *The group generated by the automaton $\mathcal{B}_4$ is the free product of 4 copies of cyclic group of order 2.*

For any $n > 4$ let $\mathcal{B}^{(n)}$ be the n-state automaton defined by the wreath recursion

$$\begin{array}{lcl} a_n & = & (c_n, b_n), \\ b_n & = & (b_n, c_n), \\ c_n & = & (q_{n1}, q_{n1})\sigma, \\ q_{n,i} & = & (q_{n,i+1}, q_{n,i+1})\sigma_{n,i}, i = 1, \ldots, n-5, \\ q_{n,n-4} & = & (d_n, d_n)\sigma_{n,n-4}, \\ d_n & = & (a_n, a_n)\sigma, \end{array}$$

where $\sigma_{n,i} \in \mathrm{Sym}(\{1, 2\})$ are chosen arbitrarily.

Theorem 0.2. *The group generated by the automaton $\mathcal{B}^{(n)}$ is the free product of n copies of cyclic group of order 2.*

The Bellaterra automaton $\mathcal{B}_3$ with 3 states is closely related to a 3-state automaton introduced by Aleshin [Ale83] in the beginning of 1980's. Namely, the only difference is in the output function, whose value is always opposite to the one of the Aleshin automaton. It was proved by M. Vorobets and Y. Vorobets [VV07] that the group generated by this automaton is a free group of rank 3.

The series of automata from [VV06] and the family of automata constructed in [SVV06] generating the free products of groups of order 2 have counterpart series and family of automata generalizing the Aleshin 3-state automaton and generating the free groups. In this paper the proofs for free products are simpler, but, as a downside, our technique does not give an answer to the question whether changing the output function of automata in our family to the opposite value produces the automata generating the free groups.

We can not miss mentioning the amusing fact that among all 190 types of 3-state automata acting on the 2-letter alphabet, that are not symmetric to each other the only automaton generating a free non-abelian group is the first Aleshin automaton (see [BGK+07]). Thus pinpointing by Aleshin this unique automaton a quarter of a century ago should really be highly valued.

The structure of the paper is as follows. All necessary definitions are given in Section 1. The automaton generating the free product of 4 cyclic groups of order 2 is studied in Section 2. In Section 3 the family of automata generating the free products of groups of order 2 is considered.

1 Preliminaries

Let X be a finite set of cardinality d. By X^* we denote the free monoid generated by X, which consists of finite words over X. This monoid can be naturally endowed with a structure of a rooted d-ary tree by declaring that v is adjacent to vx for any $v \in X^*$ and $x \in X$. The empty word corresponds to the root of the tree and X^n corresponds to the n-th level of the tree. We will be interested in the groups of automorphisms and semigroups of homomorphisms of X^*. Any such homomorphism can be defined via the notion of initial automaton.

Definition 1. *A* Mealy automaton *(or simply* automaton*) is a tuple* (Q, X, π, λ)*, where* Q *is a set (a set of states),* X *is a finite alphabet,* $\pi\colon Q \times X \to Q$ *is a transition function and* $\lambda\colon Q \times X \to X$ *is an output function. If the set of states* Q *is finite the automaton is called* finite*. If for every state* $q \in Q$ *the output function* $\lambda(q, x)$ *induces a permutation of* X*, the automaton* $\mathcal{A}$ *is called invertible. Selecting a state* $q \in Q$ *produces an* initial automaton $\mathcal{A}_q$.

Automata are often represented by the *Moore diagrams.* The Moore diagram of an automaton $\mathcal{A} = (Q, X, \pi, \lambda)$ is a directed graph in which the vertices are the states from Q and the edges have form $q \xrightarrow{x|\lambda(q,x)} \pi(q, x)$ for $q \in Q$ and $x \in X$. If the automaton is invertible, then it is common to label vertices of the Moore diagram by the permutation $\lambda(q, \cdot)$ and leave just first components from the labels of the edges. An example of Moore diagram is shown in Figure 1.

Any initial automaton induces a homomorphism of X^*. Given a word $v = x_1x_2x_3 \dots x_n \in X^*$ it scans its first letter x_1 and outputs $\lambda(x_1)$. The rest of the word is handled in a similar fashion by the initial automaton $\mathcal{A}_{\pi(x_1)}$. Formally speaking, the functions π and λ

can be extended to $\pi\colon Q\times X^*\to Q$ and $\lambda\colon Q\times X^*\to X^*$ via

$$\begin{aligned}\pi(q,x_1x_2\dots x_n)&=\pi(\pi(q,x_1),x_2x_3\dots x_n),\\ \lambda(q,x_1x_2\dots x_n)&=\lambda(q,x_1)\lambda(\pi(q,x_1),x_2x_3\dots x_n).\end{aligned}$$

By construction any initial automaton acts on X^* as a homomorphism. In case of invertible automaton it acts as an automorphism.

Definition 2. *The semigroup (group) generated by all states of automaton $\mathcal{A}$ is called an* automaton semigroup *(*automaton group*) and denoted by $\mathbb{S}(\mathcal{A})$ (respectively $\mathbb{G}(\mathcal{A})$).*

Another popular name for automaton groups and semigroups is self-similar groups and semigroups (see [Nek05]).

Conversely, any homomorphism of X^* can be encoded by the action of an initial automaton. In order to show this we need a notion of a *section* of a homomorphism at a vertex of the tree. Let g be a homomorphism of the tree X^* and $x\in X$. Then for any $v\in X^*$ we have

$$g(xv)=g(x)v'$$

for some $v'\in X^*$. Then the map $g|_x\colon X^*\to X^*$ given by

$$g|_x(v)=v'$$

defines a homomorphism of X^* and is called the *section* of g at vertex x. Furthermore, for any $x_1x_2\dots x_n\in X^*$ we define

$$g|_{x_1x_2\dots x_n}=g|_{x_1}|_{x_2}\dots|_{x_n}.$$

Given a homomorphism g of X^* we construct an initial automaton $\mathcal{A}(g)$ whose action on X^* coincides with that of g as follows. The set of states of $\mathcal{A}(g)$ is the set $\{g|_v\colon v\in X^*\}$ of different sections of g at the vertices of the tree. The transition and output functions are defined by

$$\begin{aligned}\pi(g|_v,x)&=g|_{vx},\\ \lambda(g|_v,x)&=g|_v(x).\end{aligned}$$

Throughout the paper we will use the following convention. If g and h are the elements of some (semi)group acting on set A and $a\in A$, then

$$gh(a)=h(g(a)). \tag{2}$$

Taking into account convention (2) one can compute sections of any element of an automaton semigroup as follows. If $g=g_1g_2\cdots g_n$ and $v\in X^*$, then

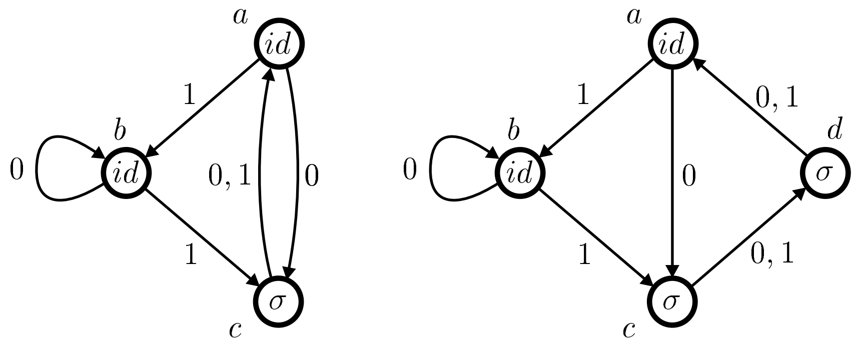

Figure 1: Bellaterra automata $\mathcal{B}_3$ and $\mathcal{B}_4$

$$g|_v = g_1|_v \cdot g_2|_{g_1(v)} \cdots g_n|_{g_1 g_2 \cdots g_{n-1}(v)}. \tag{3}$$

For any automaton group G there is a natural embedding

$$G \hookrightarrow G \wr \mathrm{Sym}(X)$$

defined by

$$G \ni g \mapsto (g_1, g_2, \ldots, g_d)\lambda(g) \in G \wr \mathrm{Sym}(X),$$

where $g_1, g_2, \ldots, g_d$ are the sections of g at the vertices of the first level, and $\lambda(g)$ is a permutation of X induced by the action of g on the first level of the tree.

The above embedding is convenient in computations involving the sections of automorphisms, as well as for defining automaton groups. For example, the group $\mathbb{G}(\mathcal{B}_3)$ generated by automaton in Figure 1, where $\sigma = (1,2)$ denotes the nontrivial element of $\mathrm{Sym}(\{1,2\})$, can be defined as

$$\begin{array}{rcl} a & = & (c, b), \\ b & = & (b, c), \\ c & = & (a, a)\sigma. \end{array} \tag{4}$$

The latter definition is sometimes called the *wreath recursion* defining the group.

For any finite automaton one can construct a *dual automaton* defined by switching the states and the alphabet as well as switching the transition and the output functions.

Definition 3. *Given a finite automaton $\mathcal{A} = (Q, X, \pi, \lambda)$ its* dual automaton $\hat{\mathcal{A}}$ *is a finite automaton $(X, Q, \hat{\lambda}, \hat{\pi})$, where*

$$\hat{\lambda}(x, q) = \lambda(q, x),$$
$$\hat{\pi}(x, q) = \pi(q, x)$$

for any $x \in X$ and $q \in Q$.

Note that the dual of the dual of an automaton $\mathcal{A}$ coincides with $\mathcal{A}$. The semigroup $\mathbb{S}(\hat{\mathcal{A}})$ generated by dual automaton $\hat{\mathcal{A}}$ of automaton $\mathcal{A}$ acts on the free monoid Q^*. This action induces the action on $\mathbb{S}(\mathcal{A})$. Similarly, $\mathbb{S}(\mathcal{A})$ acts on $\mathbb{S}(\hat{\mathcal{A}})$.

Definition 4. *For an automaton semigroup G generated by automaton $\mathcal{A}$ the* dual semigroup $\hat{G}$ *to G is a semigroup generated by a dual automaton $\hat{\mathcal{A}}$.*

A particularly important class of automata is the class of bireversible automata.

Definition 5. *An automaton $\mathcal{A}$ is called* bireversible *if it is invertible, its dual is invertible, and the dual to $\mathcal{A}^{-1}$ are invertible.*

In particular, for any group generated by a bireversible automaton $\mathcal{A}$ one can consider a dual group generated by the dual auomtaton $\hat{\mathcal{A}}$.

The following proposition is proved in [VV07] and is proved by induction on level. With a slight abuse of notations we will denote by the same symbol the element of a free monoid and its image under canonical epimorphism onto corresponding semigroup.

Proposition 1.1. *Let G be an automaton semigroup acting on X^* and generated by the finite set S. And let $\hat{G}$ be a dual semigroup to G acting on S^*. Then for any $g \in G$ and $v \in X^*$ we have $g|_v = v(g)$ in G. Similarly, for any $g \in S^*$ and $v \in \hat{G}$, $v|_g = g(v)$ in $\hat{G}$.*

2 Automaton generating $C_2 * C_2 * C_2 * C_2$

Consider the group $\mathcal{G}$ generated by the following 4-state automaton $\mathcal{B}_4$, whose transition and output functions are given by wreath recursion (its Moore diagram is shown in the right half of Figure 1)

$$\begin{array}{rcl} a & = & (c, b), \\ b & = & (b, c), \\ c & = & (d, d)\sigma, \\ d & = & (a, a)\sigma. \end{array} \tag{5}$$

The next Theorem is the main result of this section.

Theorem 2.1. *Group $\mathcal{G}$ is isomorphic to $C_2 * C_2 * C_2 * C_2$.*

The proof of this theorem is split into a number of lemmas below.

First, we note that the automaton $\mathcal{B}_4$ is bireversible. The dual group Γ to $\mathcal{G}$ is generated by the following automaton

$$\begin{array}{rcl} \mathbb{0} & = & (\mathbb{0},\mathbb{0},\mathbb{1},\mathbb{1})(a\,c\,d), \\ \mathbb{1} & = & (\mathbb{1},\mathbb{1},\mathbb{0},\mathbb{0})(a\,b\,c\,d). \end{array} \tag{6}$$

This group acts on a rooted 4-ary tree T whose vertices are labelled by the words over $\{a,b,c,d\}$. Since $a^2=(c^2,b^2)$, $b^2=(b^2,c^2)$, $c^2=(d^2,d^2)$ and $d^2=(a^2,a^2)$ we get that $a^2=b^2=c^2=d^2=1$ in Γ and the image of any word containing any of a^2, b^2, c^2 or d^2 under any element of Γ will also contain one of these subwords. Therefore there is an invariant under Γ subtree $\hat{T}$ of T consisting of all words over $\{a,b,c,d\}$ that do not have a^2, b^2, c^2 and d^2 as subwords. The root of $\hat{T}$ has 4 descendants and all the other vertices in $\hat{T}$ have three (see Figure 2, where subtree $\hat{T}$ is drawn with bold edges).

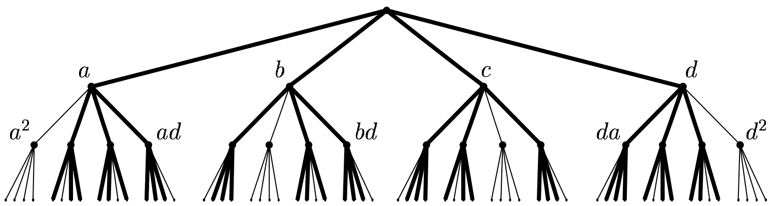

Figure 2: Trees T and $\hat{T}$

The following simple proposition was obtained independently by Z. Šunić (private communication) and the proof is implicitly contained in the book of Nekrashevych [Nek05].

Proposition 2.2. *Let G be any semigroup generated by a finite automaton and $\hat{G}$ be its dual semigroup. Then G is finite if and only if $\hat{G}$ is finite.*

Proof. Since dual of the dual of the automaton generating G coincides with this automaton, it is enough to show the implication in one direction.

Suppose G is finite. For any element $v\in\hat{G}$ and any vertex g of tree the semigroup $\hat{G}$ acts on, we have $v|_g=g(v)$ in $\hat{G}$ by Proposition 1.1.

Therefore the number of different sections of v is bounded by the size of G. But there are only finitely many different automata with a fixed number of states. Thus $\hat{G}$ is finite. $\square$

Lemma 2.3. *The group $\mathcal{G}$ is infinite.*

Proof. The lemma follows from the fact that the group acts transitively on each level of the tree. To prove this we first observe that the group $G/\operatorname{Stab}_{\mathcal{G}}(2)$ is cyclic of order 4 and the portrait of depth 2 of every element of G (rooted binary tree of depth 2, where each vertex is labelled by the permutation induced by this element at this vertex) must coincide with one of the listed in Figure 3.

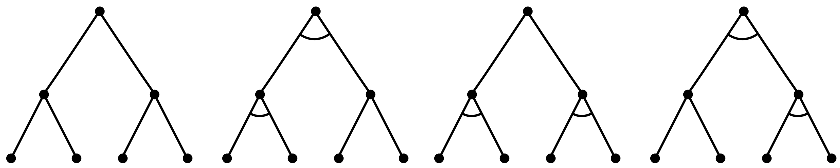

Figure 3: Possible portraits of elements of G of depth 2

It is proved in [GNS01] that an automorphism g of the rooted binary tree acts level transitively if and only if on each level the number of sections of g at the vertices of this level acting nontrivially on the first level, is odd.

By induction on level it follows that each element g of G acting nontrivially on the first level acts spherically transitively. Indeed, if the number of sections of g at the vertices of the k-th level acting nontrivially on the first level (the number of "switches" on the k-th level) is odd, then each of these sections will produce exactly one switch on the $(k+1)$-st level, while the sections acting trivially on the first level will produce either none or two switches on the $(k+1)$-st level. Thus, the total number of switches on the $(k+1)$-st level will be odd as well. $\square$

The direct corollary of Proposition 2.2 and Lemma 2.3 is

Corollary 2.4. *The group Γ is infinite.*

Corollary 2.5. *The stabilizers of levels of T in Γ are pairwise different.*

Proof. Since Γ is infinite by Corollary 2.4 and all stabilizers of levels are finite index subgroups in Γ, they are all infinite. Let $g \in \mathrm{Stab}_\Gamma(n)$ be arbitrary and nontrivial and let $m \geq n+1$ be the smallest level on which g acts nontrivially. Then there exists a vertex $v = x_1 x_2 \ldots x_{m-1}$ of the tree, such that $g|_v$ acts nontrivially on the first level. Then $g|_{x_1 x_2 \ldots x_{m-n-1}} \in \mathrm{Stab}_\Gamma(n) \setminus \mathrm{Stab}_\Gamma(n+1)$. $\square$

Lemma 2.6. *Let $\hat{T}_n$ be the subtree of $\hat{T}$ consisting of the first n levels. Then* $\mathrm{Stab}_\Gamma(n) = \mathrm{Stab}_\Gamma(\hat{T}_n)$.

Proof. Since the leaves of $\hat{T}_n$ are vertices of the n-th level of T we have $\mathrm{Stab}_\Gamma(n) \subset \mathrm{Stab}_\Gamma(T_n)$.

Suppose $v \in \mathrm{Stab}_\Gamma(\hat{T}_n) \setminus \mathrm{Stab}_\Gamma(n)$. Then there is a vertex g of the n-th level which is not in $\hat{T}$ and is not fixed under v. Since v fixes T_n it follows that $g = ftth$ and $v(g) = ftth'$ for some $f, h, h' \in G$ and $t \in \{a, b, c, d\}$. Then

$$v(fh) = v(f)v|_f(h) = f \cdot (v|_f)|_{tt}(h) = fv|_{ftt}(h) = fh'.$$

The second equality above holds since for any $t \in \{a, b, c, d\}$ we have $t^2 = 1$ and thus for any $w \in \Gamma$ by Proposition 1.1, $w|_{tt} = (tt)(w) = w$ in Γ and for any word $h \in T$ we have $w|_{tt}(h) = w(h)$.

We can repeat this procedure until we get an element of $\hat{T}_n$ not fixed under the action of v, obtaining contradiction. Thus $v \in \mathrm{Stab}_\Gamma(n) \setminus \mathrm{Stab}_\Gamma(T_n)$. $\square$

The next statement follows directly from Corollary 2.5 and Lemma 2.6.

Corollary 2.7. *For any $n \geq 1$ there is an element in Γ fixing $\hat{T}_n$ but moving some vertex in $\hat{T}_{n+1}$.*

Lemma 2.8. *The sections of any element of* $\mathrm{Stab}_\Gamma(n)$ *at the vertices of the n-th level act on the first level by even permutations.*

Proof. By self-similarity it is enough to prove the Lemma for $n = 1$. The claim follows from the fact that $|\mathrm{Stab}_\Gamma(1)/\mathrm{Stab}_\Gamma(2)| = 3^3$ (this was computed using [MS08]). Therefore the sections of any element of $\mathrm{Stab}_\Gamma(1)$ at the vertices of the first level act on the first level by permutations, whose order is a power of 3, which are either cycles of length 3 or the trivial permutation. All these are even permutations.

Below we provide a proof that does not rely on computer computations. This proof is also important because it introduces certain notation that will be used later in Section 3.

First we show that if $v \in \Gamma$ fixes vertex d, then the parities of the actions of v and $v|_d$ on the first level coincide. For this purpose we introduce a new generating set in Γ. For any $x \in \mathrm{Sym}(\{a,b,c,d\})$ denote by $\bar{x}$ the automorphism of T defined by

$$\bar{x} = (\bar{x}, \bar{x}, \bar{x}, \bar{x})x.$$

The portrait of $\bar{x}$ has x at each vertex of the tree. Since $(\mathbb{0}\mathbb{1}^{-1})^2 = (\mathbb{0}^{-1}\mathbb{1})^2 = 1$ we obtain

$$\mathbb{0}\mathbb{1}^{-1} = (\mathbb{0}\mathbb{1}^{-1}, \mathbb{0}\mathbb{1}^{-1}, \mathbb{0}\mathbb{1}^{-1}, \mathbb{0}\mathbb{1}^{-1})(a\,b) = \overline{(a\,b)} \tag{7}$$

and

$$\mathbb{0}^{-1}\mathbb{1} = (\mathbb{0}^{-1}\mathbb{1}, \mathbb{0}^{-1}\mathbb{1}, \mathbb{0}^{-1}\mathbb{1}, \mathbb{0}^{-1}\mathbb{1})(b\,c) = \overline{(b\,c)}. \tag{8}$$

This shows that $\overline{(a\,c)} \in \Gamma$. If we denote

$$\begin{array}{rcll} \alpha = \mathbb{1} \cdot \overline{(a\,c)} & = & (\alpha, \alpha, \beta, \beta) & (a\,b)(c\,d), \\ \beta = \mathbb{0} \cdot \overline{(a\,c)} & = & (\beta, \beta, \alpha, \alpha) & (c\,d), \end{array} \tag{9}$$

then $\alpha^2 = \beta^2 = 1$ and, taking into account that $\beta\alpha^{-1} = \overline{(a\,b)}$,

$$\Gamma = \langle \alpha, \beta, \overline{(b\,c)} \rangle.$$

Suppose now $v \in \Gamma$ is an arbitrary element fixing vertex d. Represent v as a word over $\{\alpha, \beta, \overline{(b\,c)}\}$

$$v = v_1 v_2 \cdots v_k,$$

then by (3)

$$v|_d = v_1|_d \cdot v_2|_{v_1(d)} \cdots v_k|_{v_1 v_2 \cdots v_{k-1}(d)}.$$

The parity of the action of v_i on the first level is different from the one of $v_i|_{v_1 v_2 \cdots v_{i-1}(d)}$ only in case v_i is α or β and $v_1 v_2 \cdots v_{i-1}(d) = c$ or $v_1 v_2 \cdots v_{i-1}(d) = d$.

Note that in this situation if $v_1 v_2 \cdots v_{i-1}(d) = c$ then $v_1 v_2 \cdots v_i(d) = d$, and if $v_1 v_2 \cdots v_{i-1}(d) = d$ then $v_1 v_2 \cdots v_i(d) = c$. The converse is also true in the following sense: if $v_1 v_2 \cdots v_{i-1}(d) \neq d$ and $v_1 v_2 \cdots v_i(d) = d$ then $v_1 v_2 \cdots v_{i-1}(d) = c$ and v_i is either α or β, and if $v_1 v_2 \cdots v_{i-1}(d) = d$ and $v_1 v_2 \cdots v_i(d) \neq d$, then $v_1 v_2 \cdots v_i(d) = c$ and v_i is either α or β. In other words, the parity of the action of v_i on the first level is different from the one of $v_i|_{v_1 v_2 \cdots v_{i-1}(d)}$ exactly when there is a change from d to anything else or from something to d in the sequence $\{d, v_1(d), \ldots, v_1 v_2 \cdots v_k(d)\}$. But since

$v_1 v_2 \cdots v_k(d) = v(d) = d$, there must be an even number of such changes. Hence, the parity is different in even number of places and the parities of the actions of v and $v|_d$ on the first level coincide.

By the above for any $g = (g|_a, g|_b, g|_c, g|_d) \in \mathrm{Stab}_\Gamma(1)$ the parity of the action of $g|_d$ on the first level is even. Furthermore, the conjugate $g^\beta = \beta^{-1} g \beta$ has decomposition

$$g^\beta = (*, *, *, (g|_c)^\alpha) \in \mathrm{Stab}_\Gamma(1),$$

which implies that $(g|_c)^\alpha$ and, hence, $g|_c$ acts on the first level by an even permutation. Finally,

$$\begin{aligned} g^{\overline{(a\,c)}} &= (*, *, (g|_a)^{\overline{(a\,c)}}, *) \in \mathrm{Stab}_\Gamma(1), \\ g^{\overline{(b\,c)}} &= (*, *, (g|_b)^{\overline{(b\,c)}}, *) \in \mathrm{Stab}_\Gamma(1). \end{aligned}$$

This shows that all sections of g at the vertices of the first level act on the first level by even permutations. □

Lemma 2.9. *The group Γ acts transitively on the levels of $\hat{T}$.*

Proof. We proceed by induction on levels. The transitivity on the first level is clear. Assume Γ acts transitively on the n-th level of $\hat{T}$. By Corollary 2.7 there is an element $v \in \Gamma$ that fixes $\hat{T}_n$ and acts nontrivially on $\hat{T}_{n+1}$. This means that there is a vertex $g \in \hat{T}_n$ such that $v(g) = g$ and $v|_g$ acts nontrivially on the first level. By Lemma 2.8 the permutation induced by $v|_g$ on the first level is even, which implies that it is a cycle of length 3. Thus $v|_g$ acts transitively on the first level of the tree.

Without loss of generality assume that g ends with d. By induction assumption, for any vertex $h_1 h_2 \dots h_{n+1}$ of $\hat{T}_{n+1}$ there is an element $w \in \Gamma$ that moves g to $h_1 h_2 \dots h_n$. Then $v^k w$, where k is 0, 1 or 2 will move ga to $h_1 h_2 \dots h_{n+1}$. Thus, Γ acts transitively on $\hat{T}_{n+1}$. □

Finally, we have all ingredients for the proof of Theorem 2.1.

Proof of Theorem 2.1. For every $n \geq 1$ there is a nontrivial element $h \in \mathcal{G}$ that belongs to the n-th level of $\hat{T}$ ($h = (ab)^{\frac{n-1}{2}} c \neq 1$ for an odd n and $h = (ab)^{\frac{n}{2}-1} ac \neq 1$ for an even n). By Lemma 2.9 the group Γ acts transitively on each level of $\hat{T}$. Therefore for any word g from the n-th level of $\hat{T}$ (which is a word of length n without double letters) there exists $v \in \Gamma$ such that

$$g|_v = v(g) = h \neq 1.$$

Thus there are no relations in $\mathcal{G}$ except $a^2 = b^2 = c^2 = d^2 = 1$. □

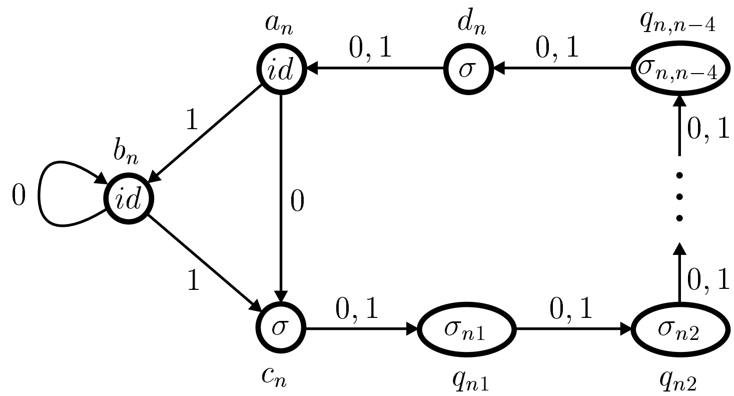

Figure 4: Bellaterra automaton $\mathcal{B}^{(n)}$

3 Family of automata generating the free products of C_2

Let us define a family of automata obtained from the automaton $\mathcal{B}_4$ by inserting new states on the path from c to d. Namely, for every integer $n > 4$ and any permutations $\sigma_{n,i} \in \mathrm{Sym}(\{1,2\})$, $i = 1, \ldots, n-4$ consider an automaton with n states $a_n, b_n, c_n, q_{n1}, q_{n2}, \ldots, q_{n,n-4}, d_n$ whose transition and output functions are given via the wreath recursion

$$
\begin{array}{lcl}
a_n & = & (c_n, b_n), \\
b_n & = & (b_n, c_n), \\
c_n & = & (q_{n1}, q_{n1})\sigma, \\
q_{n,i} & = & (q_{n,i+1}, q_{n,i+1})\sigma_{n,i}, i = 1, \ldots, n-5, \\
q_{n,n-4} & = & (d_n, d_n)\sigma_{n,n-4}, \\
d_n & = & (a_n, a_n)\sigma.
\end{array}
\tag{10}
$$

With a slight abuse of notation we denote this automaton by $\mathcal{B}^{(n)}$ regardless of the choice of permutations $\sigma_{n,i}$. The Moore diagram of $\mathcal{B}^{(n)}$ is shown in Figure 4.

From the wreath recursion it is easy to observe that the generators of $\mathcal{B}^{(n)}$ are involutions.

This section is devoted to the proof of the following theorem.

Theorem 3.1. *The group $\mathcal{G}^{(n)}$ generated by automaton $\mathcal{B}^{(n)}$ is isomorphic to the free product of n copies of the cyclic group of order 2.*

The proof relies on the results of Section 2. The approach is similar. We prove that the dual automaton acts transitively on the invariant subtree consisting of words without double letters. This yields the structure of the free product in the group $\mathcal{G}^{(n)}$.

Note that the automaton $\mathcal{B}^{(n)}$ is bireversible so that the dual group $\Gamma^{(n)}$ to $\mathcal{G}^{(n)}$ is well defined. The group $\Gamma^{(n)}$ is generated by automaton acting on the rooted n-ary tree $T^{(n)}$ as follows

$$\begin{aligned} \mathbb{0}_n &= (\mathbb{0}_n, \mathbb{0}_n, \mathbb{1}_n, \mathbb{K}_{n1}, \ldots, \mathbb{K}_{n,n-4}, \mathbb{1}_n)(a_n\, c_n\, q_{n1} \ldots q_{n,n-4}\, d_n), \\ \mathbb{1}_n &= (\mathbb{1}_n, \mathbb{1}_n, \mathbb{0}_n, \mathbb{L}_{n1}, \ldots, \mathbb{L}_{n,n-4}, \mathbb{0}_n)(a_n\, b_n\, c_n\, q_{n1} \ldots q_{n,n-4}\, d_n), \end{aligned} \tag{11}$$

where $\mathbb{K}_{n,i} = \mathbb{0}_n$ and $\mathbb{L}_{n,i} = \mathbb{1}_n$ if $\sigma_{n,i}$ is a trivial permutation, and $\mathbb{K}_{n,i} = \mathbb{1}_n$ and $\mathbb{L}_{n,i} = \mathbb{0}_n$ otherwise.

Consider a subtree $\hat{T}^{(n)}$ of $T^{(n)}$ consisting of all words over the alphabet $Y^{(n)} = \{a_n, b_n, c_n, q_{n1}, q_{n2}, \ldots, q_{n,n-4}, d_n\}$ without double letters. The root of $\hat{T}^{(n)}$ has n descendants and all other vertices have $n-1$. This subtree is invariant under the action of $\Gamma^{(n)}$.

Similarly to (7) and (8) we get that $\mathbb{0}_n \mathbb{1}_n^{-1} = \overline{(a_n\, b_n)}$ and $\mathbb{0}_n^{-1}\mathbb{1}_n = \overline{(b_n\, c_n)}$. Similarly to (9) we define transformations $\alpha_n = \mathbb{1}_n \cdot \overline{(a_n\, c_n)}$ and $\beta_n = \mathbb{0}_n \cdot \overline{(a_n\, c_n)}$ for which we have

$$\begin{aligned} \alpha_n &= (\alpha_n, \alpha_n, \beta_n, \gamma_{n1}, \ldots, \gamma_{n,n-4}, \beta_n) \quad (a_n\, b_n)(c_n\, q_{n1} \ldots q_{n,n-4}\, d_n), \\ \beta_n &= (\beta_n, \beta_n, \alpha_n, \delta_{n1}, \ldots, \delta_{n,n-4}, \alpha_n) \quad (c_n\, q_{n1} \ldots q_{n,n-4}\, d_n), \end{aligned} \tag{12}$$

where $\gamma_{n,i} = \alpha_n$ and $\delta_{n,i} = \beta_n$ if $\sigma_{n,i}$ is a trivial permutation, and $\gamma_{n,i} = \beta_n$ and $\delta_{n,i} = \alpha_n$ otherwise.

Since $\alpha_n^{-1}\beta_n = \overline{(a_n\, b_n)}$ we get a new generating set for Γ_n,

$$\Gamma^{(n)} = \langle \alpha_n, \beta_n, \overline{(b_n\, c_n)} \rangle.$$

The following lemma establishes a relation between the actions of the groups Γ and $\Gamma^{(n)}$. We consider the tree $\hat{T}$ naturally embedded in the tree $\hat{T}^{(n)}$ via a homomorphism of monoids induced by $a \mapsto a_n$, $b \mapsto b_n$, $c \mapsto c_n$, $c \mapsto c_n$, $d \mapsto d_n$. Then the group Γ acts also on $\hat{T}^{(n)}$ (the action on the letters not in the image of $\hat{T}$ is defined to be trivial).

Lemma 3.2. *For any $v \in \Gamma$ there exists $v' \in \Gamma^{(n)}$ with the following property. For any word g over $\{a_n, b_n, c_n\}$ such that $v(g)$ is also a word over $\{a_n, b_n, c_n\}$, we have $v(g) = v'(g)$.*

Proof. Let $x_1x_2\dots x_k$ be the word over $\{\alpha,\beta,\overline{(b_n\,c_n)}\}$ representing v. Define $y_i\in\{\alpha_n,\beta_n,\overline{(b_n\,c_n)}\}$ by the following rule. If $x_i=\overline{(b_n\,c_n)}$, then put $y_i=x_i$. In the case $x_i=\alpha$ (resp. $x_i=\beta$) compute the total number of α and β among $x_1,x_2,\dots,x_{i-1}$. If this number is even, then define $y_i=\alpha_n$ (resp. $y_i=\beta_n$). Otherwise, put $y_i=\alpha_n^{-1}$ (resp. $y_i=\beta_n^{-1}$).

Now let g be any word over $\{a_n,b_n,c_n\}$. We will show by induction on i that $y_1y_2\dots y_i(g)$ is obtained from $x_1x_2\dots x_i(g)$ by replacing all occurrences of d_n by q_{n1} when the total number of α and β among $x_1,x_2,\dots,x_{i-1}$ is odd, and coincides with $x_1x_2\dots x_i(g)$ otherwise.

The claim holds trivially for $i=0$. Let us prove the induction step. First of all, if $x_{i+1}=y_{i+1}=\overline{(b_n\,c_n)}$ then the relation between $y_1y_2\dots y_{i+1}(g)$ and $x_1x_2\dots x_{i+1}(g)$ is the same as between $y_1y_2\dots y_i(g)$ and $x_1x_2\dots x_i(g)$. This is because $\overline{(b_n\,c_n)}$ fixes letters d_n and $q_{n,1}$. Hence we can assume that $x_{i+1}=\alpha$ or $x_{i+1}=\beta$.

Suppose first that there is an odd number of α and β among $x_1,x_2,\dots,x_i$. By induction assumption $y_1y_2\dots y_i(g)$ is obtained from $x_1x_2\dots x_i(g)$ by replacing all occurrences of d_n by q_{n1} and, in particular, is a word over $\{a_n,b_n,c_n,q_{n1}\}$. If $x_{i+1}=\alpha$ ($x_{i+1}=\beta$), then by construction $y_{i+1}=\alpha_n^{-1}$ (respectively $y_{i+1}=\beta_n^{-1}$), for which we have

$$\begin{aligned}\alpha_n^{-1}&=(\alpha_n^{-1},\alpha_n^{-1},\beta_n^{-1},\beta_n^{-1},\gamma_{n1}^{-1},\dots,\gamma_{n,n-4}^{-1})(a_n\,b_n)(c_n\,d_n\dots q_{n1}),\\ \beta_n^{-1}&=(\alpha_n^{-1},\alpha_n^{-1},\beta_n^{-1},\beta_n^{-1},\delta_{n1}^{-1},\dots,\delta_{n,n-4}^{-1})\qquad\qquad(c_n\,d_n\dots q_{n1}).\end{aligned}\tag{13}$$

Therefore the images of $y_1y_2\dots y_i(g)$ under the actions of α_n^{-1} and β_n^{-1} coincide with the images of $x_1x_2\dots x_i(g)$ under the actions of α and β correspondingly. Thus, $y_1y_2\dots y_{i+1}(g)=x_1x_2\dots x_{i+1}(g)$, which is exactly what we need since the number of α and β among $x_1,x_2,\dots,x_{i+1}$ is even.

In case of even number of occurrences of α and β among $x_1,x_2,\dots,x_i$ by induction assumption $y_1y_2\dots y_i(g)$ coincides with $x_1x_2\dots x_i(g)$. In particular, it is a word over $\{a_n,b_n,c_n,d_n\}$. Also by construction $y_{n+1}=\alpha_n$ or $y_{n+1}=\beta_n$.

It follows from (12) that y_{i+1} acts on the letters of $y_1y_2\dots y_i(g)$ exactly as x_{i+1}, except it everywhere moves c_n to q_{n1}, instead of moving it to d_n. Therefore, the resulting word $y_1y_2\dots y_{i+1}(g)$ can be obtained from $x_1x_2\dots x_{i+1}(g)$ by changing all occurrences of d_n by q_{n1}. This agrees with the fact that the total number of α and β among $x_1,x_2,\dots,x_{i+1}$ is odd.

Finally, to finish the proof of the lemma, it is enough to put $v' = y_1 y_2 \dots y_k$ and note that if $v(g)$ is a word over $\{a_n, b_n, c_n\}$, then $v'(g)$ must coincide with $v(g)$ regardless of the number of α and β in the word representing v. $\square$

Lemma 3.3. *The group $\Gamma^{(n)}$ acts transitively on the levels of $\hat{T}^{(n)}$.*

Proof. We proceed by induction on levels. Obviously, $\Gamma^{(n)}$ acts transitively on the first level. Suppose it acts transitively on level m. We will show that any vertex of the $(m+1)$-st level can be moved to the vertex $a_n b_n a_n b_n \dots b_n a_n$ or $a_n b_n a_n b_n \dots a_n b_n$ (depending on the parity of m).

Let g be the vertex of the $(m+1)$-st level of $\hat{T}^{(n)}$. Then $g = ht$, where h is the vertex of the m-th level and $t \in Y^{(n)}$. For definiteness let us assume that m is even. By induction assumption there is $v \in \Gamma^{(n)}$ that moves h to $a_n b_n a_n b_n \dots b_n$. Then

$$v(g) = a_n b_n a_n b_n \dots a_n b_n t'$$

for some $t' \in Y^{(n)}$. Since β_n fixes $a_n b_n a_n b_n \dots a_n b_n$ and $\beta_n|_{a_n b_n a_n b_n \dots a_n b_n} = \beta_n$, after applying, if necessary, a power of β_n we can assume that $t' \in \{a_n, b_n, c_n\}$. Now we invoke the transitivity of the group Γ on $\hat{T}$. By Lemma 2.9 there is $w \in \Gamma$ such that $w(a_n b_n a_n b_n \dots a_n b_n t') = a_n b_n a_n b_n \dots a_n b_n a_n$. Then by Lemma 3.2 there is $w' \in \Gamma^{(n)}$ such that $w'(a_n b_n a_n b_n \dots a_n b_n t') = w(a_n b_n a_n b_n \dots a_n b_n t') = a_n b_n a_n b_n \dots a_n b_n a_n$. This proves transitivity of $\Gamma^{(n)}$ on the levels of $\hat{T}^{(n)}$. $\square$

Finally, Theorem 3.1 is derived from Lemma 3.3 exactly in the same way as Theorem 2.1 is obtained from Lemma 2.9.

References

[Ale72] S. V. Alešin. Finite automata and the Burnside problem for periodic groups. *Mat. Zametki*, 11:319–328, 1972.

[Ale83] S. V. Aleshin. A free group of finite automata. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (4):12–14, 1983.

[BGK+07] Ievgen Bondarenko, Rostislav Grigorchuk, Rostyslav Kravchenko, Yevgen Muntyan, Volodymyr Nekrashevych, Dmytro Savchuk, and Zoran Šunić. Classification of groups generated by 3-state automata over 2-letter alphabet. (available at *http://arxiv.org/abs/0803.3555*), 2007.

[BS98] A. M. Brunner and Said Sidki. The generation of $\mathrm{GL}(n, \mathbf{Z})$ by finite state automata. *Internat. J. Algebra Comput.*, 8(1):127–139, 1998.

[BV05] Laurent Bartholdi and Bálint Virág. Amenability via random walks. *Duke Math. J.*, 130(1):39–56, 2005. (available at *http://arxiv.org/abs/math.GR/0305262*).

[Day57] Mahlon M. Day. Amenable semigroups. *Illinois J. Math.*, 1:509–544, 1957.

[GGO07] C. K. Gupta, N. D. Gupta, and A. S. Oliynyk. Free products of finite groups acting on regular rooted trees. *Algebra Discrete Math.*, (2):91–103, 2007.

[GLSŻ00] Rostislav I. Grigorchuk, Peter Linnell, Thomas Schick, and Andrzej Żuk. On a question of Atiyah. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(9):663–668, 2000.

[Glu61] V. M. Glushkov. Abstract theory of automata. *Uspekhi mat. nauk.*, 16(5):3–62, 1961. (in Russian).

[GM05] Yair Glasner and Shahar Mozes. Automata and square complexes. *Geom. Dedicata*, 111:43–64, 2005. (available at *http://arxiv.org/abs/math.GR/0306259*).

[GNS01] Piotr W. Gawron, Volodymyr V. Nekrashevych, and Vitaly I. Sushchansky. Conjugation in tree automorphism groups. *Internat. J. Algebra Comput.*, 11(5):529–547, 2001.

[Gri80] R. I. Grigorčuk. On Burnside's problem on periodic groups. *Funktsional. Anal. i Prilozhen.*, 14(1):53–54, 1980.

[Gri84] R. I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):939–985, 1984.

[Gri98] R. I. Grigorchuk. An example of a finitely presented amenable group that does not belong to the class EG. *Mat. Sb.*, 189(1):79–100, 1998.

[GS83] Narain Gupta and Saïd Sidki. On the Burnside problem for periodic groups. *Math. Z.*, 182(3):385–388, 1983.

[GŻ02] Rostislav I. Grigorchuk and Andrzej Żuk. On a torsion-free weakly branch group defined by a three state automaton. *Internat. J. Algebra Comput.*, 12(1-2):223–246, 2002.

[Hoř63] Jiří Hořejš. Transformations defined by finite automata. *Problemy Kibernet.*, 9:23–26, 1963.

[MS08] Y. Muntyan and D. Savchuk. *AutomGrp* – `GAP` *package for computations in self-similar groups and semigroups, Version 1.1.3*, 2008. (available at *http://finautom.sourceforge.net*).

[Nek05] Volodymyr Nekrashevych. *Self-similar groups*, volume 117 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[Olī99] A. S. Olīĭnik. Free products of c_2 as groups of finitely automatic permutations. *Voprosy Algebry*, 14:158–165, 1999.

[Oli00] A. S. Oliĭnyk. Free products of finite groups and groups of finitely automatic permutations. *Tr. Mat. Inst. Steklova*, 231(Din. Sist., Avtom. i Beskon. Gruppy):323–331, 2000.

[Sus79] V. I. Sushchansky. Periodic permutation p-groups and the unrestricted Burnside problem. *DAN SSSR.*, 247(3):557–562, 1979. (in Russian).

[SVV06] B. Steinberg, M. Vorobets, and Ya. Vorobets. Automata over a binary alphabet generating free groups of even rank, 2006. (available at *http://arxiv.org/abs/math/0610033*).

[VV06] M. Vorobets and Ya. Vorobets. On a series of finite automata defining free transformation groups, 2006. (available at *http://arxiv.org/abs/math/0604328*).

[VV07] Mariya Vorobets and Yaroslav Vorobets. On a free group of transformations defined by an automaton. *Geom. Dedicata*, 124:237–249, 2007.

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