Periodic geodesics on generic translation surfaces

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Abstract

The objective of the paper is to study properties of periodic geodesics on translation surfaces that hold for generic elements of the moduli space of translation surfaces.

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1 Introduction

Let M be a compact connected oriented surface endowed with a flat metric that has a finite number of conical singularities. The flat structure on M can be determined by an atlas of coordinate charts such that all transition functions are (affine) rotations or translations in $\mathbb{R}^2$ and chart domains cover the whole surface M except for the singularities. Suppose that the atlas can be chosen so that all transition functions are translations. Then this atlas endows M with a *translation structure* and M is called a *translation surface*. Translation surfaces are closely related to Abelian differentials on compact Riemann surfaces. If x is a conical singularity of a translation surface M, then the total cone angle at this point is of the form $2\pi m$, where m is an integer. m is called the *multiplicity* of the singular point x.

A translation structure on M endows the surface punctured at its singular points with a smooth structure, a flat Riemannian metric, and an area element. Moreover, it allows us to identify the tangent space at any nonsingular point with Euclidean space $\mathbb{R}^2$ so that velocity is an invariant of the geodesic flow. In particular, each oriented geodesic is assigned a unique *direction* $v \in S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$. The study of the geodesic flow reduces to the study of the family of *directional flows* F_v, $v \in S^1$, on the surface M.

A geodesic on a translation surface cannot have self-intersections. Therefore a geodesic joining a nonsingular point to itself is necessarily closed (or *periodic*). We regard periodic geodesics as simple closed curves. The flat structure implies that any periodic geodesic belongs to a family of freely homotopic parallel periodic geodesics of the same length. The geodesics of the family fill either the whole surface or a cylindrical subset. We call this subset a *cylinder* of periodic geodesics (or a *periodic cylinder*). A periodic cylinder is bounded by geodesic segments whose endpoints are singular points. Such segments are called *saddle connections*. As a default, periodic geodesics are assumed to be unoriented but we shall consider oriented periodic geodesics and cylinders as well.

The fundamental properties of periodic geodesics on translation surfaces were established by Masur.

Theorem 1.1 ([M1], [M2], [M3]) *Let M be a translation surface.*

(a) There exists a periodic geodesic on M of length at most $c\sqrt{a}$, where a is the area of M and $c > 0$ is a constant depending only on the genus of M.

(b) The directions of periodic geodesics on M are dense in S^1.

(c) Let $N_1(M, R)$ denote the number of periodic cylinders of M of length at most $R > 0$. Then there exist $0 < c_1(M) < c_2(M) < \infty$ such that

$$c_1(M) \le N_1(M, R)/R^2 \le c_2(M)$$

for R sufficiently large.

Theorem 1.1 can be generalized as follows.

Theorem 1.2 ([Vo]) *Let M be a translation surface of area a, $m \ge 1$ be the sum of multiplicities of singular points of M, and s be the length of the shortest saddle connection of M. Then*

(a) M has a periodic cylinder of length at most $2^{2^{4m}}\sqrt{a}$ and of area at least a/m;

(b) for almost every $x \in M$ directions of periodic geodesics passing through the point x are dense in S^1;

(c) for any $R \ge 2^{2^{4m}}\sqrt{a}$,

$$\left((600m)^{(2m)^{2m}}\right)^{-1} a^{-2}s^2R^2 \le N_1(M, R) \le (400m)^{(2m)^{2m}} s^{-2}R^2.$$

Two translation surfaces M and M' are called *isomorphic* if there exists an orientation-preserving homeomorphism $f : M \to M'$ such that f maps the set of singular points of M onto the set of singular points of M' and f is a translation with respect to translation structures of M and M'. For any integers $p, n \ge 1$ let $\mathcal{MQ}(p, n)$ denote the set of equivalence classes of isomorphic translation surfaces of genus p with n singular points (of arbitrary multiplicity). $\mathcal{MQ}(p, n)$ is called the *moduli space* of such surfaces. An element of $\mathcal{MQ}(p, n)$ is a translation surface considered up to isomorphism. The moduli space $\mathcal{MQ}(p, n)$ is endowed with the structure of an affine orbifold of dimension $2(2p + n - 1)$ and with a canonical volume element (see Section 6). $\mathcal{MQ}(p, n)$ need not be connected but the number of its connected components is finite. By $\mathcal{MQ}_1(p, n)$ denote the subset of $\mathcal{MQ}(p, n)$ corresponding to translation surfaces of area 1. $\mathcal{MQ}_1(p, n)$ is a real analytic suborbifold of $\mathcal{MQ}(p, n)$ of codimension 1. The volume element on $\mathcal{MQ}(p, n)$ induces a canonical volume element on $\mathcal{MQ}_1(p, n)$ such that the volume of $\mathcal{MQ}_1(p, n)$ is finite (see [V2], [MS]). Every connected component of $\mathcal{MQ}_1(p, n)$ is of the form $\mathcal{C} \cap \mathcal{MQ}_1(p, n)$, where $\mathcal{C}$ is a connected component of $\mathcal{MQ}(p, n)$.

Suppose P is a property of translation surfaces such that P simultaneously holds or does not hold for any pair of isomorphic translation surfaces. Let $\mathcal{C}$ be a nonempty open subset of $\mathcal{MQ}(p, n)$ or $\mathcal{MQ}_1(p, n)$. We say that the property P is *generic* for translation surfaces in $\mathcal{C}$ or that P holds for a *generic* $M \in \mathcal{C}$ if P holds for almost all $M \in \mathcal{C}$ with respect to the volume element on $\mathcal{MQ}(p,n)$ (resp. $\mathcal{MQ}_1(p,n)$).

The simplest property of periodic geodesics on generic translation surfaces that is not enjoyed by all translation surfaces concerns regularity of periodic cylinders. A periodic cylinder is called *regular* if it is bounded by saddle connections of the same length as geodesics in the cylinder.

Proposition 1.3 ([EM]) *Generic translation surfaces admit only regular periodic cylinders.*

Two nonintersecting periodic geodesics are called *homologous* if they break the translation surface into two components. Homologous geodesics are parallel and of the same length. Two periodic geodesics in the same cylinder are obviously homologous but the converse is not true. As shown by Eskin, Masur, and Zorich [EMZ], for any positive integer k a generic translation surface of sufficiently high genus admits k distinct cylinders of homologous periodic geodesics.

Proposition 1.4 *(a) For a generic translation surface any two nonhomologous periodic geodesics are of different length and direction.*

(b) For a generic translation surface $M \in \mathcal{MQ}(p,n)$, $(p,n) \neq (1,1)$, all periodic cylinders are of different area.

The group $\mathrm{SL}(2,\mathbb{R})$ acts on the set of translation surfaces of genus p with n singular points by postcomposition of the chart maps with linear transformations from $\mathrm{SL}(2,\mathbb{R})$. This action descends to an action on the moduli space $\mathcal{MQ}(p,n)$, which is affine and leaves invariant the subspace $\mathcal{MQ}_1(p,n)$. The $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{MQ}_1(p,n)$ is volume preserving and ergodic on each connected component of $\mathcal{MQ}_1(p,n)$ (see [V1]). This fact is basic in establishing genericity of many properties of translation surfaces, in particular, the properties stated below.

For any translation surface M and any $R > 0$ let $N_1(M,R)$ denote the number of periodic cylinders of M of length at most R. By $N_2(M,R)$ denote the sum of areas of these cylinders. Further, for any $x \in M$ let $N_3(M,x,R)$ denote the number of periodic geodesics on M of length at most R that pass through the point x. For any $\sigma \geq 0$ let $N_4(M,\sigma,R)$ denote the number of periodic cylinders of M of length at most R and of area greater than σ.

Let $\mathcal{C}$ be a connected component of the space $\mathcal{MQ}_1(p,n)$.

Theorem 1.5 ([EM]) *For a generic translation surface $M \in \mathcal{C}$,*

$$\lim_{R\to\infty} N_1(M,R)/R^2 = c_1(\mathcal{C}),$$

where $c_1(\mathcal{C}) > 0$ depends only on the component $\mathcal{C}$.

Theorem 1.5 was proved by Eskin and Masur [EM] who sharpened results of Veech [V3]. The constants $c_1(\mathcal{C})$ were found by Eskin, Masur, and Zorich [EMZ]. It turns out that explicit evaluation of these constants involves a lot of calculation.

Theorem 1.6 *(a) For a generic translation surface $M \in \mathcal{C}$,*

$$\lim_{R\to\infty} N_2(M,R)/R^2 = c_2(\mathcal{C}),$$

where $c_2(\mathcal{C}) > 0$ depends only on the component $\mathcal{C}$.

(b) $c_2(\mathcal{C}) = c_1(\mathcal{C})/m_{\mathcal{C}}$, where $m_{\mathcal{C}} = 2p-2+n$ is the sum of multiplicities of singular points for translation surfaces in $\mathcal{C}$.

Theorem 1.7 *For a generic translation surface $M \in \mathcal{C}$,*

$$\lim_{R\to\infty} N_3(M,x,R)/R^2 = c_2(\mathcal{C})$$

for almost every $x \in M$.

The following theorem answers, in particular, Question 13.4 of the paper [V3].

Theorem 1.8 *For a generic translation surface $M \in \mathcal{C}$,*

$$\lim_{R\to\infty} N_4(M,\sigma,R)/R^2 = (1-\sigma)^{m_{\mathcal{C}}-1}c_1(\mathcal{C})$$

for all $\sigma \in [0,1)$.

For every translation surface M of area 1 we define three families $\alpha_{M,R}$, $\delta_{M,R}$, and $D_{M,R}$ of measures depending on the parameter $R > 0$. $\alpha_{M,R}$ is a measure on $[0,1]$; for any $K \subset [0,1]$ let $\alpha_{M,R}(K)$ be the number of periodic cylinders of M of length at most R and of area in the set K. $\delta_{M,R}$ is a measure on the unit circle S^1; for any $U \subset S^1$ let $\delta_{M,R}(U)$ be the number of oriented periodic cylinders of length at most R with directions in the set U. $D_{M,R}$ is a measure on $S^1 \times [0,1]$; for any $U \subset S^1 \times [0,1]$ let $D_{M,R}(U)$ be the number of oriented periodic cylinders of length at most R with direction v and area a such that $(v,a) \in U$. Let R_M be the length of the shortest periodic geodesic on M. For any $R \geq R_M$ the measures $\alpha_{M,R}$, $\delta_{M,R}$, and $D_{M,R}$ are nonzero so we define probability measures $\tilde{\alpha}_{M,R} = (\alpha_{M,R}([0,1]))^{-1}\alpha_{M,R}$, $\tilde{\delta}_{M,R} = (\delta_{M,R}(S^1))^{-1}\delta_{M,R}$, and $\widetilde{D}_{M,R} = (D_{M,R}(S^1 \times [0,1]))^{-1}D_{M,R}$. The measures $\tilde{\alpha}_{M,R}$ describe the distribution of areas of periodic cylinders on M. The measures $\tilde{\delta}_{M,R}$ describe the distribution of their directions. The measures $\widetilde{D}_{M,R}$ describe the joint distribution of directions and areas.

For any integer $m \geq 1$ let λ_m denote a unique Borel measure on $[0,1]$ such that $\lambda_m([\sigma,1]) = (1-\sigma)^{m-1}$ for all $\sigma \in [0,1)$. Let $\mathfrak{m}_1$ denote Lebesgue measure on S^1 normalized so that $\mathfrak{m}_1(S^1) = 1$.

Theorem 1.9 *For a generic translation surface $M \in \mathcal{C}$ we have the following weak convergence of measures:*

$$\lim_{R\to\infty} \tilde{\alpha}_{M,R} = \lambda_{m_{\mathcal{C}}},$$

$$\lim_{R\to\infty} \tilde{\delta}_{M,R} = \mathfrak{m}_1,$$

$$\lim_{R\to\infty} \widetilde{D}_{M,R} = \mathfrak{m}_1 \times \lambda_{m_{\mathcal{C}}}.$$

Theorem 1.9 means that for a generic translation surface $M \in \mathcal{C}$ the directions of periodic geodesics on M are uniformly distributed in S^1 while the areas of periodic cylinders are distributed according to the measure $\lambda_{m_{\mathcal{C}}}$. Moreover, the distributions of directions and areas of periodic cylinders are independent. In view of Theorems 1.5 and 1.6(a), the ratio $c_2(\mathcal{C})/c_1(\mathcal{C})$ may be regarded as the *mean area* of periodic cylinders on a generic translation surface $M \in \mathcal{C}$. Indeed, $c_2(\mathcal{C})/c_1(\mathcal{C}) = 1/m_{\mathcal{C}}$ is the expectation of a random variable taking values in $[0,1]$ with probabilities given by the measure $\lambda_{m_{\mathcal{C}}}$.

For any translation surface M and any $x \in M$ we define a family $\delta_{M,x,R}$ of measures on S^1 depending on $R > 0$. For any $U \in S^1$ let $\delta_{M,x,R}(U)$ be the number of oriented periodic geodesics on M of length at most R passing through the point x in directions from U. If the measure $\delta_{M,x,R}$ is nonzero then we define a probability measure $\tilde{\delta}_{M,x,R} = (\delta_{M,x,R}(S^1))^{-1}\delta_{M,x,R}$.

Theorem 1.10 *For a generic translation surface $M \in \mathcal{C}$ the weak convergence of measures*

$$\lim_{R\to\infty} \tilde{\delta}_{M,x,R} = \mathfrak{m}_1$$

takes place for almost all $x \in M$.

Thus the directions of periodic geodesics passing through a generic point on a generic translation surface are uniformly distributed in S^1.

The paper is organized as follows. In Section 2 we review Veech's theory of Siegel measures. In Section 3 this theory is applied to a general counting problem related to the growth of the number of periodic geodesics. The tools to treat limit distributions are developed in Section 4. Section 5 contains preliminaries on translation surfaces and Delaunay partitions. In Section 6 we consider moduli spaces of translation surfaces. Section 7 is devoted to proofs of main results of the paper. Some proofs rely on estimates of volumes of certain subsets in the moduli space of translation surfaces. These estimates are obtained in Section 8.

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2 Siegel measures

Let $\mathfrak{M}$ denote the set of locally finite Borel measures on $\mathbb{R}^2$. Given a bounded, compactly supported Borel function ψ on $\mathbb{R}^2$, set

$$\hat{\psi}(\nu) = \int_{\mathbb{R}^2} \psi(x)\, d\nu(x)$$

for any $\nu \in \mathfrak{M}$. Let $C_c(\mathbb{R}^2)$ denote the space of continuous, compactly supported functions on $\mathbb{R}^2$. Endow $\mathfrak{M}$ with the $C_c(\mathbb{R}^2)$ weak-$*$ topology. By definition, this is the smallest topology such that $\hat{\psi}$ is continuous on $\mathfrak{M}$ when $\psi \in C_c(\mathbb{R}^2)$.

For any $R > 0$ let $B(R)$ denote the disk of radius R in $\mathbb{R}^2$ centered at the origin. For any $\nu \in \mathfrak{M}$ and any $R > 0$ set $N_\nu(R) = \nu(B(R))$. The function N_ν is called the *growth function* of the measure ν. By $\mathfrak{M}_2$ denote the set of measures $\nu \in \mathfrak{M}$ such that $M(\nu) < \infty$, where

$$M(\nu) = \sup_{R>0} \frac{N_\nu(R)}{R^2}.$$

As a topological space, $\mathfrak{M}_2$ is a countable union of metrizable compacta. Namely, for any $c > 0$ the subspace $\mathfrak{M}_2(c) = \{\nu \in \mathfrak{M}_2 \mid M(\nu) \le c\}$ is compact and metrizable. A measure $\nu \in \mathfrak{M}$ is called *even* if $\nu(U) = \nu(-U)$ for any Borel set $U \subset \mathbb{R}^2$. The set $\mathfrak{M}^e$ of all even measures is a closed subset of $\mathfrak{M}$. By $\mathfrak{m}$ denote Lebesgue measure on $\mathbb{R}^2$. Clearly, $\mathfrak{m} \in \mathfrak{M}_2(\pi) \subset \mathfrak{M}_2$ and $\mathfrak{m} \in \mathfrak{M}^e$.

Let $g \in \mathrm{SL}(2,\mathbb{R})$. We regard g as a linear transformation of $\mathbb{R}^2$. For any $\nu \in \mathfrak{M}$ define a measure $g\nu$ by $g\nu(U) = \nu(g^{-1}(U))$, $U \subset \mathbb{R}^2$ a Borel set. The map $\mathrm{SL}(2,\mathbb{R}) \times \mathfrak{M} \ni (g,\nu) \mapsto g\nu$ defines a continuous action of the group $\mathrm{SL}(2,\mathbb{R})$ on $\mathfrak{M}$. The sets $\mathfrak{M}_2$ and $\mathfrak{M}^e$ are invariant under this action.

For any $R > 0$ define a transformation $T_R : \mathfrak{M} \to \mathfrak{M}$ by $T_R\nu(U) = R^{-2}\nu(RU)$, $\nu \in \mathfrak{M}$, $U \subset \mathbb{R}^2$ a Borel set. The family $\{T_R\}_{R>0}$ defines a continuous action of the group $\mathbb{R}^+$ on $\mathfrak{M}$. The sets $\mathfrak{M}_2$, $\mathfrak{M}^e$, and each of the subsets $\mathfrak{M}_2(c)$, $c > 0$, are invariant under this action. Besides, the action of $\mathbb{R}^+$ on $\mathfrak{M}$ commutes with the action of $\mathrm{SL}(2,\mathbb{R})$.

Denote by $\mathcal{P}(\mathfrak{M}_2)$ the set of Borel probability measures on $\mathfrak{M}_2$. An element $\mu \in \mathcal{P}(\mathfrak{M}_2)$ is called a *Siegel measure* if the $\mathrm{SL}(2,\mathbb{R})$ action on $\mathfrak{M}_2$ leaves μ invariant and is ergodic with respect to μ. Siegel measures were introduced and studied by Veech [V3].

Theorem 2.1 ([V3]) *Assume $\mu \in \mathcal{P}(\mathfrak{M}_2)$ is a Siegel measure. Then there exists $c(\mu) \ge 0$ such that for any compactly supported bounded Borel function ψ on $\mathbb{R}^2$, the function $\hat{\psi}$ belongs to $L^1(\mathfrak{M}_2,\mu)$ and*

$$\int_{\mathfrak{M}_2} \hat{\psi}(\nu)\, d\mu(\nu) = c(\mu) \int_{\mathbb{R}^2} \psi(x)\, d\mathfrak{m}(x).$$

The number $c(\mu)$ is called the *Siegel-Veech constant* of the measure μ.

Theorem 2.2 ([V3]) *Assume $\mu \in \mathcal{P}(\mathfrak{M}_2)$ is a Siegel measure. Then*

$$\lim_{R\to\infty} \int_{\mathfrak{M}_2} \Big| R^{-2} N_\nu(R) - c(\mu)\pi \Big|\, d\mu(\nu) = 0.$$

As a consequence, there exists a sequence $R_n \to \infty$ such that for μ-a.e. $\nu \in \mathfrak{M}_2$,

$$\lim_{n\to\infty} \frac{N_\nu(R_n)}{R_n^2} = c(\mu)\pi.$$

Let $\mathfrak{M}_2^e = \mathfrak{M}_2 \cap \mathfrak{M}^e$. Denote by $\mathcal{P}(\mathfrak{M}_2^e)$ the set of measures $\mu \in \mathcal{P}(\mathfrak{M}_2)$ supported on $\mathfrak{M}_2^e$.

Theorem 2.3 ([V3]) *Assume $\mu \in \mathcal{P}(\mathfrak{M}_2^e)$ is a Siegel measure. Then there exists a sequence $R_n \to \infty$ such that for μ-a.e. $\nu \in \mathfrak{M}_2^e$,*

$$\lim_{n\to\infty} T_{R_n}\nu = c(\mu)\mathfrak{m},$$

where convergence is in the $C_c(\mathbb{R}^2)$ weak-$*$ topology.

Theorem 2.3 is derived from Theorem 2.2 by applying the following criterion.

Theorem 2.4 ([V3]) *Let $\{\nu_\alpha \mid a \in A\}$ be a net of even, locally finite Borel measures on $\mathbb{R}^2$. Assume there exist $c < \infty$ and a dense set $F \subset \mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$ such that for any $(g,t) \in F$,*

$$\lim_{\alpha\in A} \nu_\alpha(tgB(1)) = \pi t^2 c.$$

Then

$$\lim_{\alpha\in A} \int_{\mathbb{R}^2} \psi(x)\, d\nu_\alpha(x) = c \int_{\mathbb{R}^2} \psi(x)\, d\mathfrak{m}(x)$$

for any $\psi \in C_c(\mathbb{R}^2)$, i.e., $\lim_{\alpha\in A} \nu_\alpha = c\mathfrak{m}$ in $\mathfrak{M}$.

Let $\mu \in \mathcal{P}(\mathfrak{M}_2)$ be a Siegel measure. Set $L^{1+}(\mathfrak{M}_2,\mu) = \bigcup_{\epsilon>0} L^{1+\epsilon}(\mathfrak{M}_2,\mu)$. The measure μ is called *regular* if $\hat{\psi} \in L^{1+}(\mathfrak{M}_2,\mu)$ for any $\psi \in C_c(\mathbb{R}^2)$.

Theorem 2.5 *If $\mu \in \mathcal{P}(\mathfrak{M}_2)$ is a regular Siegel measure, then for μ-a.e. $\nu \in \mathfrak{M}_2$,*

$$\lim_{R\to\infty} \frac{N_\nu(R)}{R^2} = c(\mu)\pi.$$

A closely related result, which is reproduced below as Part II of Theorem 3.2, was proved by Eskin and Masur [EM]. The proof of Theorem 2.5 is almost the same and will be omitted.

Theorem 2.6 *If $\mu \in \mathcal{P}(\mathfrak{M}_2^e)$ is a regular Siegel measure, then for μ-a.e. $\nu \in \mathfrak{M}_2^e$,*

$$\lim_{R\to\infty} T_R\nu = c(\mu)\mathfrak{m}.$$

Proof. Let $c(\mu)$ be the Siegel-Veech constant of the measure μ. By Theorem 2.5, there exists a Borel set $U \subset \mathfrak{M}_2^e$ such that $\mu(U) = 1$ and for all $\nu \in U$,

$$\lim_{R\to\infty} R^{-2} N_\nu(R) = c(\mu)\pi.$$

Suppose G is a countable dense subset of the group $\mathrm{SL}(2,\mathbb{R})$. Set $U_0 = \bigcap_{g\in G} gU$. Clearly, U_0 is a Borel subset of $\mathfrak{M}_2^e$ and $\mu(U_0) = 1$. Take a measure $\nu \in U_0$. For any $g \in G$ and any $t > 0$ we have

$$T_R\nu(tgB(1)) = R^{-2}\nu(gB(Rt)) = R^{-2}(g^{-1}\nu)(B(Rt)) = R^{-2}N_{g^{-1}\nu}(Rt).$$

Since $g^{-1}\nu \in U$, it follows that $\lim_{R\to\infty} T_R\nu(tgB(1)) = t^2c(\mu)\pi$. By Theorem 2.4, $\lim_{R\to\infty} T_R\nu = c(\mu)\mathfrak{m}$ in $\mathfrak{M}$. ∎

3 Counting problem

Let V denote a sequence $v_1, v_2, \dots$ of vectors in $\mathbb{R}^2$ equipped with a sequence $w_1, w_2, \dots$ of positive reals. The number w_k is called the *weight* of the vector v_k. It is assumed that the sequence of vectors tends to infinity or is finite (possibly empty). By $\mathcal{V}$ denote the set of all such sequences of vectors with weights. Two elements $V_1, V_2 \in \mathcal{V}$ are considered to be equal if one of them can be obtained from the other by rearranging its vectors along with the corresponding rearrangement of weights. The group $\mathrm{SL}(2,\mathbb{R})$ acts on the set $\mathcal{V}$ by the natural action on vectors and the trivial action on weights. To each $V \in \mathcal{V}$ we assign a linear functional $\Phi[V]$ on the space of compactly supported functions on $\mathbb{R}^2$; the functional is defined by the relation $\Phi[V](\psi) = \sum_{k=1}^{\infty} w_k \psi(v_k)$. Furthermore, for any $R > 0$ set $N_V(R) = \Phi[V](\chi_{B(R)}) = \sum_{k:|v_k|\le R} w_k$. The function N_V is called the *growth function* of V. If all weights of V are equal to 1, then $N_V(R)$ counts the number of vectors of V in the disk $B(R)$.

Given $V \in \mathcal{V}$, let $v_1, v_2, \dots$ be the sequence of vectors of V and $w_1, w_2, \dots$ be the sequence of weights. For any Borel set $U \subset \mathbb{R}^2$ let $\nu_V(U) = \sum_{k:v_k \in U} w_k$. It is easy to see that ν_V is a Borel measure on $\mathbb{R}^2$. Since the sequence $v_1, v_2, \dots$ either tends to infinity or is finite, the measure ν_V is locally finite, i.e., $\nu_V \in \mathfrak{M}$. The growth function of V coincides with the growth function of the measure ν_V. For any compactly supported Borel function ψ on $\mathbb{R}^2$,

$$\Phi[V](\psi) = \int_{\mathbb{R}^2} \psi(x)\, d\nu_V(x).$$

Let $\mathcal{M}$ be a locally compact metric space endowed with a finite nonzero Borel measure μ. Suppose the group $\mathrm{SL}(2,\mathbb{R})$ acts on the space $\mathcal{M}$ by homeomorphisms. We assume that the measure μ is invariant under this action and the action is *ergodic*, that is, any measurable subset of $\mathcal{M}$ invariant under the action is of zero or full measure. In what follows we consider maps $V : \mathcal{M} \to \mathcal{V}$ satisfying all or at least some of the following conditions:

(0) for any $\psi \in C_c(\mathbb{R}^2)$ the function $\mathcal{M} \ni \omega \mapsto \Phi[V(\omega)](\psi)$ is Borel;

(A) the map V intertwines the actions of the group $\mathrm{SL}(2,\mathbb{R})$ on the spaces $\mathcal{M}$ and $\mathcal{V}$, that is, $V(g\omega) = gV(\omega)$ for any $g \in \mathrm{SL}(2,\mathbb{R})$ and any $\omega \in \mathcal{M}$;

(B) for any $\omega \in \mathcal{M}$ there exists $c(\omega) < \infty$ such that $N_{V(\omega)}(R) \le c(\omega)R^2$ for all $R > 0$;

(B′) for any compact subset $K \subset \mathcal{M}$ there exists $c(K) < \infty$ such that $N_{V(\omega)}(R) \le c(\omega)R^2$ for all $\omega \in K$ and $R > 0$;

(C) there exists $R_0 > 0$ such that the function $\omega \mapsto N_{V(\omega)}(R_0)$ belongs to the space $L^{1+}(\mathcal{M},\mu) = \bigcup_{\epsilon>0} L^{1+\epsilon}(\mathcal{M},\mu)$;

(C′) for any $R > 0$ the function $\omega \mapsto N_{V(\omega)}(R)$ belongs to $L^{1+}(\mathcal{M},\mu)$;

(C$''$) for any $\psi \in C_c(\mathbb{R}^2)$ the function $\omega \mapsto \Phi[V(\omega)](\psi)$ belongs to $L^{1+}(\mathcal{M}, \mu)$;

(E) $V(\omega) = -V(\omega)$ for any $\omega \in \mathcal{M}$.

Note that conditions (C), (C$'$), and (C$''$) depend on the measure μ.

Lemma 3.1 *Suppose a map $V : \mathcal{M} \to \mathcal{V}$ satisfies condition (0). Then conditions (A) and (C) imply condition (C$'$), while conditions (C$'$) and (C$''$) are equivalent.*

Proof. Given a bounded, compactly supported Borel function ψ on $\mathbb{R}^2$, set $\tilde{\psi}(\omega) = \Phi[V(\omega)](\psi)$ for any $\omega \in \mathcal{M}$. For any $R > 0$ there exists a sequence $\psi_1, \psi_2, \dots$ of functions in $C_c(\mathbb{R}^2)$ such that $\psi_1 \geq \psi_2 \geq \dots$ and $\psi_n \to \chi_{B(R)}$ pointwise as $n \to \infty$. It follows that $\tilde{\psi}_1 \geq \tilde{\psi}_2 \geq \dots$ and $\tilde{\psi}_n \to \tilde{\chi}_{B(R)}$ pointwise as $n \to \infty$. Condition (0) implies $\tilde{\psi}_1, \tilde{\psi}_2, \dots$ are Borel functions on $\mathcal{M}$. Then $\tilde{\chi}_{B(R)}$ is a Borel function as well.

Suppose conditions (A) and (C) hold for the map V. Condition (C) means that $\tilde{\chi}_{B(R_0)} \in L^{1+}(\mathcal{M}, \mu)$ for some $R_0 > 0$. Condition (A) implies $\tilde{\chi}_{gB(R_0)}(\omega) = \tilde{\chi}_{B(R_0)}(g^{-1}\omega)$ for all $g \in \mathrm{SL}(2, \mathbb{R})$ and $\omega \in \mathcal{M}$. It follows that $\tilde{\chi}_{gB(R_0)} \in L^{1+}(\mathcal{M}, \mu)$ for any $g \in \mathrm{SL}(2, \mathbb{R})$. Since $\mathrm{SL}(2, \mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{(0, 0)\}$, for any $x \in \mathbb{R}^2$ there exist a neighborhood U_x of x and an operator $g_x \in \mathrm{SL}(2, \mathbb{R})$ such that $g_x U_x \subset B(R_0)$. Given $R > 0$, the disk $B(R)$ is covered by finitely many neighborhoods $U_{x_1}, U_{x_2}, \dots, U_{x_k}$. Then $B(R) \subset \bigcup_{i=1}^k h_i B(R_0)$, where $h_i = g_{x_i}^{-1}$. It follows that $\tilde{\chi}_{B(R)} \leq \sum_{i=1}^k \tilde{\chi}_{h_i B(R_0)}$. By the above the functions $\tilde{\chi}_{h_1 B(R_0)}, \dots, \tilde{\chi}_{h_k B(R_0)}$ belongs to $L^{1+}(\mathcal{M}, \mu)$. Since $\tilde{\chi}_{B(R)}$ is a nonnegative Borel function, it belongs to $L^{1+}(\mathcal{M}, \mu)$ as well. Thus, $\tilde{\chi}_{B(R)} \in L^{1+}(\mathcal{M}, \mu)$ for any $R > 0$, i.e., condition (C$'$) holds.

For any $\psi \in C_c(\mathbb{R}^2)$ there exist $c, R > 0$ such that $|\psi| \leq c\chi_{B(R)}$. Then $|\tilde{\psi}| \leq c\tilde{\chi}_{B(R)}$. It follows that $\tilde{\psi} \in L^{1+}(\mathcal{M}, \mu)$ whenever $\tilde{\chi}_{B(R)} \in L^{1+}(\mathcal{M}, \mu)$. Thus condition (C$'$) implies condition (C$''$). On the other hand, for any $R > 0$ there exists $\psi \in C_c(\mathbb{R}^2)$ such that $\chi_{B(R)} \leq \psi$. Then $0 \leq \tilde{\chi}_{B(R)} \leq \tilde{\psi}$. By the above $\tilde{\chi}_{B(R)}$ is a Borel function, hence $\tilde{\chi}_{B(R)} \in L^{1+}(\mathcal{M}, \mu)$ whenever $\tilde{\psi} \in L^{1+}(\mathcal{M}, \mu)$. Thus condition (C$''$) implies condition (C$'$). ■

Theorem 3.2 *Let $V : \mathcal{M} \to \mathcal{V}$ be a map satisfying condition (0).*

I. *Suppose the map V satisfies conditions (A) and (B). Then there exists $c_{V,\mu} \geq 0$ such that for any compactly supported bounded Borel function ψ on $\mathbb{R}^2$, the function $\omega \mapsto \Phi[V(\omega)](\psi)$ belongs to $L^1(\mathcal{M}, \mu)$ and*

$$\frac{1}{\mu(\mathcal{M})} \int_{\mathcal{M}} \Phi[V(\omega)](\psi)\, d\mu(\omega) = c_{V,\mu} \int_{\mathbb{R}^2} \psi(x)\, d\mathfrak{m}(x).$$

II. *If V satisfies conditions (A), (B), and (C), then for μ-a.e. $\omega \in \mathcal{M}$,*

$$\lim_{R \to \infty} \frac{N_{V(\omega)}(R)}{R^2} = \pi c_{V,\mu}.$$

III. *If V satisfies conditions (A), (B), (C), and (E), then for μ-a.e. $\omega \in \mathcal{M}$,*

$$\lim_{R \to \infty} \int_{\mathbb{R}^2} \frac{1}{R^2} \psi\left(\frac{x}{R}\right) d\nu_{V(\omega)}(x) = c_{V,\mu} \int_{\mathbb{R}^2} \psi(x)\, d\mathfrak{m}(x)$$

for each compactly supported, Riemann integrable function $\psi : \mathbb{R}^2 \to \mathbb{R}$. In particular, $\lim_{R\to\infty} T_R \nu_{V(\omega)} = c_{V,\mu}\mathfrak{m}$ in $\mathfrak{M}$ for μ-a.e. $\omega \in \mathcal{M}$.

The constant $c_{V,\mu}$ is called the *Siegel-Veech constant* of the pair (V, μ).

Part I of Theorem 3.2 was proved by Veech [V3] in some particular cases. The proof in the general case requires no changes. Parts I and II was proved by Eskin and Masur [EM] assuming conditions (A), (B′), (C), and, implicitly, condition (0) hold. In fact, the weaker condition (B) was used in the proof instead of condition (B′). Although condition (C) is not necessary for Part I to be true, it simplifies the proof significantly. Besides, condition (C) allows one to prove Part I without using ergodicity of the $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{M}$.

Proof of Theorem 3.2. We shall derive Parts I–III of the theorem from Theorems 2.1, 2.5, and 2.6, respectively.

Define a map $F : \mathcal{M} \to \mathfrak{M}$ by $F(\omega) = \nu_{V(\omega)}$, $\omega \in \mathcal{M}$. It is easy to observe that F is a Borel map if and only if condition (0) holds for the map V. Further, $F(\mathcal{M}) \subset \mathfrak{M}_2$ if and only if V satisfies condition (B). If condition (A) holds, then $F(g\omega) = gF(\omega)$ for all $g \in \mathrm{SL}(2,\mathbb{R})$ and $\omega \in \mathcal{M}$. If condition (E) holds, then $F(\mathcal{M}) \subset \mathfrak{M}^e$.

Suppose the map V satisfies conditions (0), (A), and (B). For any Borel subset $U \subset \mathfrak{M}_2$ let $\mu_V(U) = \mu(F^{-1}(U))/\mu(\mathcal{M})$. Condition (0) implies μ_V is a well-defined finite Borel measure on $\mathfrak{M}_2$. Condition (B) implies μ_V is a probability measure. Since μ is invariant under the $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{M}$, condition (A) implies μ_V is invariant under the $\mathrm{SL}(2,\mathbb{R})$ action on $\mathfrak{M}_2$. Ergodicity of the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{M}$ with respect to μ implies ergodicity of the action on $\mathfrak{M}_2$ with respect to μ_V. Thus, μ_V is a Siegel measure. By Theorem 2.1, for any compactly supported bounded Borel function $\psi : \mathbb{R}^2 \to \mathbb{R}$ the function $\hat{\psi}$ belongs to $L^1(\mathfrak{M}_2, \mu_V)$ and

$$\int_{\mathfrak{M}_2} \hat{\psi}(\nu)\, d\mu_V(\nu) = c_{V,\mu} \int_{\mathbb{R}^2} \psi(x)\, d\mathfrak{m}(x),$$

where $c_{V,\mu}$ is the Siegel-Veech constant of the measure μ_V. Then $\hat{\psi} \circ F \in L^1(\mathcal{M}, \mu)$ and

$$\frac{1}{\mu(\mathcal{M})} \int_{\mathcal{M}} \hat{\psi}(F(\omega))\, d\mu(\omega) = \int_{\mathfrak{M}_2} \hat{\psi}(\nu)\, d\mu_V(\nu).$$

As $\hat{\psi}(F(\omega)) = \Phi[V(\omega)](\psi)$ for any $\omega \in \mathcal{M}$, Part I of the theorem follows.

Now suppose V satisfies conditions (0), (A), (B), and (C). By Lemma 3.1, condition (C″) also holds for V. Condition (C″) means that $\hat{\psi} \circ F \in L^{1+}(\mathcal{M}, \mu)$ for any $\psi \in C_c(\mathbb{R}^2)$. It follows that $\hat{\psi} \in L^{1+}(\mathfrak{M}_2, \mu_V)$ for any $\psi \in C_c(\mathbb{R}^2)$. Therefore μ_V is a regular Siegel measure. By Theorem 2.5,

$$\lim_{R\to\infty} R^{-2} N_\nu(R) = \pi c_{V,\mu}$$

for μ_V-a.e. $\nu \in \mathfrak{M}_2$, or, equivalently,

$$\lim_{R\to\infty} R^{-2} N_{F(\omega)}(R) = \pi c_{V,\mu}$$

for μ-a.e. $\omega \in \mathcal{M}$. As for any $\omega \in \mathcal{M}$ the growth function $N_{F(\omega)}$ of the measure $F(\omega) = \nu_{V(\omega)}$ coincides with the growth function $N_{V(\omega)}$, Part II of the theorem follows.

Now suppose V satisfies conditions (0), (A), (B), (C), and (E). By the above μ_V is a regular Siegel measure. Condition (E) implies $\mu_V \in \mathcal{P}(\mathfrak{M}_2^e)$. By Theorem 2.6,

$$\lim_{R\to\infty} T_R\nu = c_{V,\mu}\mathfrak{m}$$

for μ_V-a.e. $\nu \in \mathfrak{M}_2^e$. It follows that

$$\lim_{R\to\infty} T_R\nu_{V(\omega)} = c_{V,\mu}\mathfrak{m}$$

for $\omega \in U$, where $U \subset \mathcal{M}$ is a Borel set such that $\mu(U) = \mu(\mathcal{M})$. Given any compactly supported, Riemann integrable function $\psi : \mathbb{R}^2 \to \mathbb{R}$, there exist two sequences $\psi_1^+, \psi_2^+, \dots$ and $\psi_1^-, \psi_2^-, \dots$ of functions in $C_c(\mathbb{R}^2)$ such that $\psi_n^- \le \psi \le \psi_n^+$ and $\int(\psi_n^+ - \psi_n^-)\,d\mathfrak{m} < 1/n$ for $n = 1, 2, \dots$. Obviously,

$$\int_{\mathbb{R}^2} R^{-2}\psi_n^-(x/R)\,d\nu_{V(\omega)} \le \int_{\mathbb{R}^2} R^{-2}\psi(x/R)\,d\nu_{V(\omega)} \le \int_{\mathbb{R}^2} R^{-2}\psi_n^+(x/R)\,d\nu_{V(\omega)}$$

for any $\omega \in \mathcal{M}$, and

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\psi_n^+\,d\mathfrak{m} = \lim_{n\to\infty}\int_{\mathbb{R}^2}\psi_n^-\,d\mathfrak{m} = \int_{\mathbb{R}^2}\psi\,d\mathfrak{m}.$$

If $\omega \in U$, then

$$\int_{\mathbb{R}^2} R^{-2}\psi_n^+(x/R)\,d\nu_{V(\omega)} = c_{V,\mu}\int_{\mathbb{R}^2}\psi_n^+\,d\mathfrak{m},$$
$$\int_{\mathbb{R}^2} R^{-2}\psi_n^-(x/R)\,d\nu_{V(\omega)} = c_{V,\mu}\int_{\mathbb{R}^2}\psi_n^-\,d\mathfrak{m}$$

for $n = 1, 2, \dots$. It follows that

$$\int_{\mathbb{R}^2} R^{-2}\psi(x/R)\,d\nu_{V(\omega)} = c_{V,\mu}\int_{\mathbb{R}^2}\psi\,d\mathfrak{m}$$

for any $\omega \in U$. Part III of the theorem is proved. ∎

Proposition 3.3 *Let $V : \mathcal{M} \to \mathcal{V}$ be a map satisfying conditions (0), (A), and (B). Let $c_{V,\mu}$ denote the Siegel-Veech constant of the pair (V, μ). Then $c_{V,\mu} = 0$ if and only if the sequence $V(\omega)$ is empty for μ-a.e. $\omega \in \mathcal{M}$.*

Proof. By Part I of Theorem 3.2,

$$\frac{1}{\mu(\mathcal{M})}\int_{\mathcal{M}}\Phi[V(\omega)](\chi_{B(R)})\,d\mu(\omega) = \pi c_{V,\mu}R^2$$

for any $R > 0$. Observe that $\Phi[V(\omega)](\chi_{B(R)}) \ge 0$, and $\Phi[V(\omega)](\chi_{B(R)}) = 0$ if and only if the sequence $V(\omega)$ contains no vectors of length at most R. Clearly, if $V(\omega)$ is empty for μ-a.e. $\omega \in \mathcal{M}$, then $c_{V,\mu} = 0$. Conversely, if $c_{V,\mu} = 0$, then for any $R > 0$ we have $\Phi[V(\omega)](\chi_{B(R)}) = 0$ for μ-a.e. $\omega \in \mathcal{M}$. Hence there exists a Borel set $U \subset \mathcal{M}$ such that $\mu(U) = \mu(\mathcal{M})$ and $\Phi[V(\omega)](\chi_{B(1)}) = \Phi[V(\omega)](\chi_{B(2)}) = \ldots = 0$ for $\omega \in U$. Then for any $\omega \in U$ the sequence $V(\omega)$ is empty. ∎

4 Limit distributions

Denote by $\mathcal{V}_1$ the subset of $\mathcal{V}$ consisting of sequences with all weights equal to 1. For any $K \subset \mathbb{R}^+ = (0, \infty)$ define a map $W_K : \mathcal{V} \to \mathcal{V}_1$ as follows. Given $V \in \mathcal{V}$, the sequence $W_K V$ is obtained from V by removing all vectors with weights outside K and then changing weights of the remaining vectors to 1.

Let $V \in \mathcal{V}$, $v_1, v_2, \dots$ be the sequence of vectors of V, and $w_1, w_2, \dots$ be the sequence of weights. For any compactly supported function ψ on $\mathbb{R}^2$ and any function f on $\mathbb{R}^+$ set $\Psi[V](f,\psi) = \sum_{k\geq 1} f(w_k)\psi(v_k)$. Clearly, $\Phi[V](\psi) = \Psi[V](\mathrm{id}, \psi)$ and $\Phi[W_K V](\psi) = \Psi[V](\chi_K, \psi)$ for any $K \subset \mathbb{R}^+$. By $C_c(\mathbb{R}^+)$ denote the space of continuous, compactly supported functions on $\mathbb{R}^+$. Note that any function $f \in C_c(\mathbb{R}^+)$ vanishes in a neighborhood of 0. $\Psi[V]$ is a bilinear functional on $C_c(\mathbb{R}^+) \times C_c(\mathbb{R}^2)$.

Furthermore, we associate to V three families $\alpha_{V,R}$, $\delta_{V,R}$, and $D_{V,R}$ of measures depending on the parameter $R > 0$. $\alpha_{V,R}$ is a measure on $\mathbb{R}^+$; for any $K \subset \mathbb{R}^+$ let $\alpha_{V,R}(K)$ be the number of indices k such that $|v_k| \leq R$ and $w_k \in K$. $\delta_{V,R}$ is a measure on the unit circle $S^1 = \{v \in \mathbb{R}^2 : |v| = 1\}$; for any $U \subset S^1$ let $\delta_{V,R}(U)$ be the number of indices k such that $0 < |v_k| \leq R$ and $v_k/|v_k| \in U$. Finally, $D_{V,R}$ is a measure on $S^1 \times \mathbb{R}^+$; for any $U \subset S^1 \times \mathbb{R}^+$ let $D_{V,R}(U)$ be the number of indices k such that $0 < |v_k| \leq R$ and $(v_k/|v_k|, w_k) \in U$. The measures $\alpha_{V,R}$, $R > 0$, describe the distribution of weights of vectors in V. The measures $\delta_{V,R}$, $R > 0$, describe the distribution of their directions. The measures $D_{V,R}$, $R > 0$, describe the joint distribution of directions and weights.

Suppose V contains nonzero vectors and denote by R_0 the length of the shortest nonzero vector in V. For any $R \geq R_0$ the measures $\alpha_{V,R}$, $\delta_{V,R}$, and $D_{V,R}$ are nonzero so we can define probability measures $\tilde{\alpha}_{V,R} = (\alpha_{V,R}(\mathbb{R}^+))^{-1}\alpha_{V,R}$, $\tilde{\delta}_{V,R} = (\delta_{V,R}(S^1))^{-1}\delta_{V,R}$, and $\widetilde{D}_{V,R} = (D_{V,R}(S^1 \times \mathbb{R}^+))^{-1} D_{V,R}$.

Now let V be a map of the space $\mathcal{M}$ introduced in Section 3 to $\mathcal{V}$. We add two more conditions to the list started in Section 3:

($0'$) for any $a > 0$ the map $\mathcal{M} \ni \omega \mapsto W_{(a,\infty)}V(\omega)$ satisfies condition (0);

($0''$) for any $f \in C_c(\mathbb{R}^+)$ and $\psi \in C_c(\mathbb{R}^2)$ the function $\mathcal{M} \ni \omega \mapsto \Psi[V(\omega)](f,\psi)$ is Borel.

Lemma 4.1 *Suppose a map $V : \mathcal{M} \to \mathcal{V}$ satisfies condition ($0'$). Then for any Borel set $K \subset \mathbb{R}^+$ the map $W_K V$ satisfies condition (0).*

Proof. Let $\mathcal{K}$ denote the set of subsets $K \subset \mathbb{R}^+$ such that the map $W_K V$ satisfies condition (0). We shall verify the following properties of $\mathcal{K}$:

(i) if $A, B \in \mathcal{K}$ and $A \cap B = \emptyset$, then $A \cup B \in \mathcal{K}$;

(ii) if $A, B \in \mathcal{K}$ and $B \subset A$, then $A \setminus B \in \mathcal{K}$;

(iii) if $A_1, A_2, \ldots \in \mathcal{K}$ and $\lim\limits_{n\to\infty} A_n$ exists, then $\lim\limits_{n\to\infty} A_n \in \mathcal{K}$.

Let $A, B \subset \mathbb{R}^+$. If $A \cap B = \emptyset$, then $\Phi[W_{A\cup B}V(\omega)] = \Phi[W_A V(\omega)] + \Phi[W_B V(\omega)]$ for all $\omega \in \mathcal{M}$. This implies property (i). If $B \subset A$, then $\Phi[W_{A\setminus B}V(\omega)] = \Phi[W_A V(\omega)] - \Phi[W_B V(\omega)]$ for all $\omega \in \mathcal{M}$. This implies property (ii). Now suppose $A_1, A_2, \dots$ are subsets of $\mathbb{R}^+$ such that the set $A = \lim_{n\to\infty} A_n$ exists. Then $\chi_A(t) = \lim_{n\to\infty} \chi_{A_n}(t)$ for all $t > 0$. It follows that $\Phi[W_A V(\omega)](\psi) = \lim_{n\to\infty} \Phi[W_{A_n} V(\omega)](\psi)$ for any $\psi \in C_c(\mathbb{R}^2)$ and $\omega \in \mathcal{M}$. This implies property (iii).

Condition ($0'$) means that $(a, \infty) \in \mathcal{K}$ for any $a > 0$. Property (iii) implies $\mathbb{R}^+ = \bigcup_{n=1}^\infty (n^{-1}, \infty) \in \mathcal{K}$ and $\emptyset = \bigcap_{n=1}^\infty (n, \infty) \in \mathcal{K}$. For any a, b, $0 \le a < b$, we have $(a, b] = (a, \infty) \setminus (b, \infty) \in \mathcal{K}$. Further, $(a, b) = \bigcup_{n=1}^\infty (a, b - n^{-1}] \in \mathcal{K}$. Since any open subset of $\mathbb{R}^+$ is a disjoint union of intervals, properties (i) and (iii) imply $\mathcal{K}$ contains all open subsets of $\mathbb{R}^+$. Then it follows from the properties (i), (ii), and (iii) that $\mathcal{K}$ contains all Borel subsets of $\mathbb{R}^+$. $\blacksquare$

Lemma 4.2 *Conditions ($0'$) and ($0''$) are equivalent. Condition ($0''$) implies condition (0).*

Proof. Let $\mathcal{F}$ denote the set of functions f on $\mathbb{R}^+$ such that the function $\mathcal{M} \ni \omega \mapsto \Psi[V(\omega)](f, \psi)$ is Borel for any $\psi \in C_c(\mathbb{R}^2)$. $\mathcal{F}$ have the following properties:

(i) if $f_1, f_2 \in \mathcal{F}$, then $a_1 f_1 + a_2 f_2 \in \mathcal{F}$ for any $a_1, a_2 \in \mathbb{R}$;

(ii) if $f_1, f_2, \dots \in \mathcal{F}$ and $f = \lim\limits_{n\to\infty} f_n$ pointwise, then $f \in \mathcal{F}$.

The first property follows from the fact that $\Psi[V(\omega)](f, \psi)$ depends linearly on f. To verify the second property observe that $\Psi[V(\omega)](f, \psi) = \lim_{n\to\infty} \Psi[V(\omega)](f_n, \psi)$ for any $\psi \in C_c(\mathbb{R}^2)$ and $\omega \in \mathcal{M}$ if $f(t) = \lim_{n\to\infty} f_n(t)$ for all $t > 0$.

Suppose the map V satisfies condition ($0'$). Then Lemma 4.1 implies $\chi_K \in \mathcal{F}$ for all Borel sets $K \subset \mathbb{R}^+$. Given $f \in C_c(\mathbb{R}^+)$, for any $\epsilon > 0$ there exist Borel sets $K_1, \dots, K_n$ and constants $a_1, \dots, a_n$ such that $\sup |f - (a_1\chi_{K_1} + \cdots + a_n \chi_{K_n})| < \epsilon$. It follows from properties (i) and (ii) that $f \in \mathcal{F}$. Thus $C_c(\mathbb{R}^+) \subset \mathcal{F}$, i.e., condition ($0''$) holds. Conversely, if the map V satisfies condition ($0''$), that is, $C_c(\mathbb{R}^+) \subset \mathcal{F}$, then it easily follows from property (ii) that $\chi_{(a,\infty)} \in \mathcal{F}$ for any $a > 0$ and $\mathrm{id}_{(0,\infty)} \in \mathcal{F}$. This means that conditions ($0'$) and (0) hold for V. $\blacksquare$

Proposition 4.3 *Let a map $V : \mathcal{M} \to \mathcal{V}$ satisfy conditions ($0'$), (A), and (B). Then (a) for any Borel set $K \subset \mathbb{R}^+$ the map $W_K V$ satisfies conditions (0) and (A);*

(b) for any $K \subset (\epsilon, \infty)$, $\epsilon > 0$, the map $W_K V$ satisfies condition (B);

(c) there exists a unique Borel measure $\lambda_{V,\mu}$ on $\mathbb{R}^+$ with the following property: for any Borel set $K \subset \mathbb{R}^+$ such that $W_K V$ satisfies conditions (0), (A), and (B) the Siegel-Veech constant of the pair $(W_K V, \mu)$ is equal to $\lambda_{V,\mu}(K)$;

(d) if $c_{V,\mu}$ denotes the Siegel-Veech constant of the pair (V, μ), then

$$c_{V,\mu} = \int_0^\infty t \, d\lambda_{V,\mu}(t);$$

(e) if condition (B) holds for the map $W_{(0,\infty)}V$, then the measure $\lambda_{V,\mu}$ is finite and condition (B) holds for $W_K V$ for any $K \subset \mathbb{R}^+$.

The measure $\lambda_{V,\mu}$ is called the *Siegel-Veech measure* of the pair (V, μ).

Proof of Proposition 4.3. By Lemma 4.1, the map $W_K V$ satisfies condition (0) for any Borel set $K \subset \mathbb{R}^+$. For any $K \subset \mathbb{R}^+$ condition (A) holds for $W_K V$ since the $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{V}$ does not affect weights. If $K \subset (\epsilon, \infty)$, $\epsilon > 0$, then $N_{W_K V(\omega)}(R) \le \epsilon^{-1} N_{V(\omega)}(R)$ for all $\omega \in \mathcal{M}$ and $R > 0$, hence condition (B) holds for $W_K V$. If condition (B) holds for the map $W_{(0,\infty)} V$, then this condition holds for $W_K V$ for any $K \subset \mathbb{R}^+$ since $N_{W_K V(\omega)}(R) \le N_{W_{(0,\infty)} V(\omega)}(R)$ for all $\omega \in \mathcal{M}$ and $R > 0$.

Let $K \subset \mathbb{R}^+$ be a Borel set. By the above for any $\psi \in C_c(\mathbb{R}^2)$ the function $\mathcal{M} \ni \omega \mapsto \Phi[W_K V(\omega)](\psi)$ is Borel. Then it follows that for any $R > 0$ the function $\omega \mapsto \Phi[W_K V(\omega)](\chi_{B(R)}) = N_{W_K V(\omega)}(R)$ is Borel (cf. the proof of Lemma 3.1). Set

$$\lambda_{V,\mu}(K) = \frac{1}{\pi \cdot \mu(\mathcal{M})} \int_{\mathcal{M}} N_{W_K V(\omega)}(1)\, d\mu(\omega).$$

Obviously, $0 \le \lambda_{V,\mu}(K) \le \infty$. Suppose $K_1, K_2, \dots$ are disjoint Borel subsets of $\mathbb{R}^+$ and denote $K = \bigcup_{n=1}^{\infty} K_n$. Then $N_{W_K V(\omega)}(1) = \sum_{n=1}^{\infty} N_{W_{K_n} V(\omega)}(1)$ for any $\omega \in \mathcal{M}$. It follows that $\lambda_{V,\mu}(K) = \sum_{n=1}^{\infty} \lambda_{V,\mu}(K_n)$. Thus $\lambda_{V,\mu}$ is a measure. If the map $W_K V$ satisfies conditions (0), (A), and (B) for some Borel set $K \subset \mathbb{R}^+$, then the Siegel-Veech constant of $(W_K V, \mu)$ is equal to $\lambda_{V,\mu}(K)$ due to Part I of Theorem 3.2. If condition (B) holds for the map $W_{(0,\infty)} V$, then $\lambda_{V,\mu}(\mathbb{R}^+)$ is the Siegel-Veech constant of $(W_{(0,\infty)} V, \mu)$, in particular, $\lambda_{V,\mu}(\mathbb{R}^+) < \infty$.

Suppose $\tilde{\lambda}$ is a Borel measure on $\mathbb{R}^+$ having the property required in the statement (c). Take any Borel set $K \subset \mathbb{R}^+$. For each $\epsilon > 0$ let $K_\epsilon = K \cap (\epsilon, \infty)$. By the above conditions (0), (A), and (B) hold for the map $W_{K_\epsilon} V$, hence $\tilde{\lambda}(K_\epsilon)$ is the Siegel-Veech constant of $(W_{K_\epsilon} V, \mu)$, in particular, $\tilde{\lambda}(K_\epsilon) = \lambda_{V,\mu}(K_\epsilon)$. Then $\tilde{\lambda}(K) = \lim_{\epsilon \to 0} \tilde{\lambda}(K_\epsilon) = \lim_{\epsilon \to 0} \lambda_{V,\mu}(K_\epsilon) = \lambda_{V,\mu}(K)$. Thus $\tilde{\lambda} = \lambda_{V,\mu}$.

Let $f_1, f_2, \dots$ be the functions on $\mathbb{R}^+$ such that $f_n(t) = 2^{-n}k$ for $2^{-n}k < t \le 2^{-n}(k+1)$, $k = 0, 1, \dots$. Then $0 \le f_1 \le f_2 \le \dots$ and $\lim_{n\to\infty} f_n(t) = t$ for all $t > 0$. It follows that

$$\lim_{n\to\infty} \int_0^\infty f_n(t)\, d\lambda_{V,\mu}(t) = \int_0^\infty t\, d\lambda_{V,\mu}(t),$$

$$\lim_{n\to\infty} \int_{\mathcal{M}} \Psi[V(\omega)](f_n, \chi_{B(1)})\, d\mu(\omega) = \int_{\mathcal{M}} N_{V(\omega)}(1)\, d\mu(\omega).$$

For any integer $n \ge 1$ we have

$$\int_{\mathcal{M}} \Psi[V(\omega)](f_n, \chi_{B(1)})\, d\mu(\omega) = \sum_{k=1}^{\infty} 2^{-n}k \int_{\mathcal{M}} N_{W_{(2^{-n}k, 2^{-n}(k+1)]} V(\omega)}(1)\, d\mu(\omega) =$$

$$\pi \cdot \mu(\mathcal{M}) \sum_{k=1}^{\infty} 2^{-n}k \cdot \lambda_{V,\mu}((2^{-n}k, 2^{-n}(k+1)]) = \pi \cdot \mu(\mathcal{M}) \int_0^\infty f_n(t)\, d\lambda_{V,\mu}(t),$$

hence

$$\int_0^\infty t\, d\lambda_{V,\mu}(t) = \frac{1}{\pi \cdot \mu(\mathcal{M})} \int_{\mathcal{M}} N_{V(\omega)}(1)\, d\mu(\omega).$$

By Part I of Theorem 3.2, the right-hand side of the last equality is equal to the Siegel-Veech constant of the pair (V, μ). ∎

Proposition 4.4 *Suppose a map $V : \mathcal{M} \to \mathcal{V}$ satisfies conditions (0′), (A), (B), and (C). Then for μ-a.e. $\omega \in \mathcal{M}$,*

$$\lim_{R\to\infty} (\pi R^2)^{-1} \alpha_{V(\omega),R} = \lambda_{V,\mu}$$

in the following sense:

$$\lim_{R\to\infty} \frac{1}{\pi R^2} \int_0^\infty f(t)\, d\alpha_{V(\omega),R}(t) = \int_0^\infty f(t)\, d\lambda_{V,\mu}(t) \tag{1}$$

for any bounded function $f \in C(\mathbb{R}^+)$ vanishing in a neighborhood of 0.

If, in addition, condition (B) holds for the map $W_{(0,\infty)}V$ and the Siegel-Veech constant of the pair (V, μ) is nonzero, then for μ-a.e. $\omega \in \mathcal{M}$,

$$\lim_{R\to\infty} \tilde{\alpha}_{V(\omega),R} = (\lambda_{V,\mu}(\mathbb{R}^+))^{-1} \lambda_{V,\mu}$$

in the following sense:

$$\lim_{R\to\infty} \int_0^\infty f(t)\, d\tilde{\alpha}_{V(\omega),R}(t) = \frac{1}{\lambda_{V,\mu}(\mathbb{R}^+)} \int_0^\infty f(t)\, d\lambda_{V,\mu}(t) \tag{2}$$

for any bounded function $f \in C(\mathbb{R}^+)$.

Proof. Let $K \subset \mathbb{R}^+$ be a Borel set such that $K \subset (\epsilon, \infty)$ for some $\epsilon > 0$. By Proposition 4.3, the map $W_K V$ satisfies conditions (0), (A), and (B). Condition (C) also holds for $W_K V$ since $N_{W_K V(\omega)}(R) \le \epsilon^{-1} N_{V(\omega)}(R)$ for any $\omega \in \mathcal{M}$ and $R > 0$. By Part II of Theorem 3.2, for μ-a.e. $\omega \in \mathcal{M}$ we have

$$\lim_{R\to\infty} R^{-2} N_{W_K V(\omega)}(R) = \pi \lambda_{V,\mu}(K). \tag{3}$$

Let $\mathcal{K}$ denote the countable collection of sets consisting of intervals $(r_1, r_2]$ such that r_1, r_2 are rational numbers and $0 < r_1 < r_2$. By the above there exists a Borel set $U \subset \mathcal{M}$, $\mu(U) = \mu(\mathcal{M})$, such that for any $\omega \in U$ and any $K \in \mathcal{K}$ relation (3) holds.

Take any $\omega \in U$. For each function $f \in L^1(\mathbb{R}^+, \lambda_{V,\mu})$ set

$$F_\omega(f) = \limsup_{R\to\infty} \left| \frac{1}{\pi R^2} \int_0^\infty f(t)\, d\alpha_{V(\omega),R}(t) - \int_0^\infty f(t)\, d\lambda_{V,\mu}(t) \right|.$$

Suppose $f \in C_c(\mathbb{R}^+)$ and choose $a > 1$ such that $f = 0$ outside the segment $[1/a, a]$. For any $\epsilon > 0$ there exist disjoint sets $K_1, \dots, K_n \in \mathcal{K}$ and constants $b_1, \dots, b_n$ such that the function $f_\epsilon = b_1 \chi_{K_1} + \dots + b_n \chi_{K_n}$ satisfies the inequality $\sup |f - f_\epsilon| < \epsilon$. Observe that

$$\int_0^\infty f_\epsilon(t)\, d\alpha_{V(\omega),R}(t) = \sum_{i=1}^n b_i N_{W_{K_i} V(\omega)}(R),$$

$$\int_0^\infty f_\epsilon(t)\,d\lambda_{V,\mu}(t) = \sum_{i=1}^n b_i \lambda_{V,\mu}(K_i).$$

Since $\omega \in U$, it follows that $F_\omega(f_\epsilon) = 0$. Further,

$$\int_0^\infty |f(t) - f_\epsilon(t)|\,d\alpha_{V(\omega),R}(t) \le \epsilon N_{W_{[1/a,a]}V(\omega)}(R),$$

$$\int_0^\infty |f(t) - f_\epsilon(t)|\,d\lambda_{V,\mu}(t) \le \epsilon \lambda_{V,\mu}([a^{-1}, a]).$$

It follows that $F_\omega(f) \le \epsilon M(a,\omega)$, where $M(a,\omega) = \sup_{R>0} R^{-2} N_{W_{[1/a,a]}V(\omega)}(R) + \lambda_{V,\mu}([1/a, a])$. Note that $M(a,\omega) < \infty$ while ϵ can be chosen arbitrarily small. Thus $F_\omega(f) = 0$.

Now let f be a bounded continuous function on $\mathbb{R}^+$ vanishing in a neighborhood of 0. For any $a > 0$ there exists $\tilde{f}_a \in C_c(\mathbb{R}^+)$ such that $\tilde{f}_a = f$ on $(0,a)$ and $\sup|\tilde{f}_a - f| \le \sup|f|$. We have

$$\int_0^\infty |f(t) - \tilde{f}_a(t)|\,d\alpha_{V(\omega),R}(t) \le N_{W_{[a,\infty)}V(\omega)}(R) \sup|f|,$$

$$\int_0^\infty |f(t) - \tilde{f}_a(t)|\,d\lambda_{V,\mu}(t) \le \lambda_{V,\mu}([a,\infty)) \sup|f|.$$

By the above $F_\omega(\tilde{f}_a) = 0$, hence $F_\omega(f) \le M_\infty(a,\omega)\sup|f|$, where $M_\infty(a,\omega) = \sup_{R>0} R^{-2} N_{W_{[a,\infty)}V(\omega)}(R) + \lambda_{V,\mu}([a,\infty))$. Since $N_{W_{[a,\infty)}V(\omega)}(R) \le a^{-1} N_{V(\omega)}(R)$, it follows that $M_\infty(a,\omega) \to 0$ as $a \to \infty$. Thus $F_\omega(f) = 0$, i.e., equality (1) holds.

Now suppose the map $W_{(0,\infty)}V$ satisfies condition (B). Then $\lambda_{V,\mu}$ is a finite measure. Let $\mathcal{K}_0$ denote the collection of intervals $(r_1, r_2]$ such that r_1, r_2 are rational numbers and $0 \le r_1 < r_2$. Then there exists a Borel set $U_0 \subset U$, $\mu(U_0) = \mu(\mathcal{M})$, such that for any $\omega \in U_0$ and any $K \in \mathcal{K}_0$ relation (3) holds. Take any $\omega \in U_0$. Let f be a bounded continuous function on $\mathbb{R}^+$. For any $\epsilon > 0$ there exists a bounded function $\hat{f}_\epsilon \in C(\mathbb{R}^+)$ such that $\hat{f}_\epsilon$ vanishes in a neighborhood of 0, $\hat{f}_\epsilon = f$ on (ϵ, ∞), and $\sup|\hat{f}_\epsilon - f| \le \sup|f|$. We have

$$\int_0^\infty |f(t) - \hat{f}_\epsilon(t)|\,d\alpha_{V(\omega),R}(t) \le N_{W_{(0,\epsilon]}V(\omega)}(R) \sup|f|,$$

$$\int_0^\infty |f(t) - \hat{f}_\epsilon(t)|\,d\lambda_{V,\mu}(t) \le \lambda_{V,\mu}((0,\epsilon]) \sup|f|.$$

By the above $F_\omega(\hat{f}_\epsilon) = 0$, therefore $F_\omega(f) \le M_0(\epsilon,\omega)\sup|f|$, where $M_0(\epsilon,\omega) = \limsup_{R>0} (\pi R^2)^{-1} N_{W_{(0,\epsilon]}V(\omega)}(R) + \lambda_{V,\mu}((0,\epsilon])$. Since $\omega \in U_0$, we have $M_0(\epsilon,\omega) \le \lambda_{V,\mu}((0,r]) + \lambda_{V,\mu}((0,\epsilon])$ for any rational $r \ge \epsilon$. Since $\lambda_{V,\mu}(\mathbb{R}^+) < \infty$, it follows that $M_0(\epsilon,\omega) \le 2\lambda_{V,\mu}((0,\epsilon])$ and $M_0(\epsilon,\omega) \to 0$ as $\epsilon \to 0$. Thus $F_\omega(f) = 0$, i.e., equality (1) holds for f. In particular, equality (1) holds for $f = 1$ so we get

$$\lim_{R\to\infty} (\pi R^2)^{-1} \alpha_{V(\omega),R}(\mathbb{R}^+) = \lambda_{V,\mu}(\mathbb{R}^+).$$

If the Siegel-Veech constant of (V,μ) is nonzero, then $\lambda_{V,\mu}(\mathbb{R}^+) > 0$ and equality (2) follows. ∎

Proposition 4.5 *Suppose a map $V : \mathcal{M} \to \mathcal{V}_1$ satisfies conditions (0), (A), (B), (C), (E), and the Siegel-Veech constant of the pair (V, μ) is nonzero. Then for μ-a.e. $\omega \in \mathcal{M}$,*

$$\lim_{R\to\infty} \tilde{\delta}_{V(\omega),R} = \mathfrak{m}_1,$$

where $\mathfrak{m}_1$ is Lebesgue measure on S^1 normalized so that $\mathfrak{m}_1(S^1) = 1$. The convergence means that

$$\lim_{R\to\infty} \int_{S^1} \phi(\theta)\, d\tilde{\delta}_{V(\omega),R}(\theta) = \int_{S^1} \phi(\theta)\, d\mathfrak{m}_1(\theta) \tag{4}$$

for any Riemann integrable function ϕ on S^1.

Proof. By Part III of Theorem 3.2, there exists a Borel set $U \subset \mathcal{M}$, $\mu(U) = \mu(\mathcal{M})$, such that for any $\omega \in U$,

$$\lim_{R\to\infty} \int_{\mathbb{R}^2} \frac{1}{R^2}\psi\Big(\frac{x}{R}\Big)\, d\nu_{V(\omega)}(x) = c_{V,\mu} \int_{\mathbb{R}^2} \psi(x)\, d\mathfrak{m}(x) \tag{5}$$

for each compactly supported, Riemann integrable function ψ on $\mathbb{R}^2$. Here $c_{V,\mu}$ is the Siegel-Veech constant of (V, μ).

Take any $\omega \in U$. Let ϕ be an arbitrary Riemann integrable function on S^1. For any $x \in \mathbb{R}^2$ set $\phi_1(x) = \phi(x/|x|)$ if $0 < |x| \le 1$, and $\phi_1(x) = 0$ otherwise. Then ϕ_1 is a Riemann integrable function on $\mathbb{R}^2$, therefore equality (5) holds for $\psi = \phi_1$. Since $V(\mathcal{M}) \subset \mathcal{V}_1$, we have

$$\int_{\mathbb{R}^2} \phi_1(x/R)\, d\nu_{V(\omega)}(x) = \int_{S^1} \phi(\theta)\, d\delta_{V(\omega),R}(\theta)$$

for all $R > 0$. Besides,

$$\int_{\mathbb{R}^2} \phi_1(x)\, d\mathfrak{m}(x) = \pi \int_{S^1} \phi(\theta)\, d\mathfrak{m}_1(\theta).$$

Hence

$$\lim_{R\to\infty} R^{-2} \int_{S^1} \phi(\theta)\, d\delta_{V(\omega),R}(\theta) = \pi c_{V,\mu} \int_{S^1} \phi(\theta)\, d\mathfrak{m}_1(\theta).$$

In particular, the latter relation holds for $\phi = 1$ so we get

$$\lim_{R\to\infty} R^{-2}\delta_{V(\omega),R}(S^1) = \pi c_{V,\mu}.$$

Since $c_{V,\mu} > 0$, equality (4) follows. ∎

Proposition 4.6 *Suppose a map $V : \mathcal{M} \to \mathcal{V}$ satisfies conditions (0′), (A), (B), (C), and (E). Then for μ-a.e. $\omega \in \mathcal{M}$,*

$$\lim_{R\to\infty} (\pi R^2)^{-1} D_{V(\omega),R} = \mathfrak{m}_1 \times \lambda_{V,\mu}$$

in the following sense:

$$\lim_{R\to\infty} \frac{1}{\pi R^2} \int_{S^1\times\mathbb{R}^+} f(\Theta)\, dD_{V(\omega),R}(\Theta) = \int_{S^1}\int_0^\infty f(\theta,t)\, d\mathfrak{m}_1(\theta)\, d\lambda_{V,\mu}(t) \tag{6}$$

for any bounded function $f \in C(S^1 \times \mathbb{R}^+)$ vanishing in a neighborhood of $S^1 \times \{0\}$.

If, in addition, condition (B) holds for the map $W_{(0,\infty)}V$ and the Siegel-Veech constant of the pair (V,μ) is nonzero, then for μ-a.e. $\omega \in \mathcal{M}$,

$$\lim_{R\to\infty} \widetilde{D}_{V(\omega),R} = (\lambda_{V,\mu}(\mathbb{R}^+))^{-1}\mathfrak{m}_1 \times \lambda_{V,\mu}$$

in the following sense:

$$\lim_{R\to\infty} \int_{S^1\times\mathbb{R}^+} f(\Theta)\, d\widetilde{D}_{V(\omega),R}(\Theta) = \frac{1}{\lambda_{V,\mu}(\mathbb{R}^+)} \int_{S^1}\int_0^\infty f(\theta,t)\, d\mathfrak{m}_1(\theta)\, d\lambda_{V,\mu}(t)$$

for any bounded function $f \in C(S^1 \times \mathbb{R}^+)$.

Proof. Let $K \subset \mathbb{R}^+$ be a Borel set such that $K \subset (\epsilon,\infty)$ for some $\epsilon > 0$. Then the map $W_K V$ satisfies conditions (0), (A), (B), (C), and (E) (cf. the proof of Proposition 4.4). As shown in the proof of Proposition 4.5, for μ-a.e. $\omega \in \mathcal{M}$ we have

$$\lim_{R\to\infty} R^{-2} \int_{S^1} \phi(\theta)\, d\delta_{W_K V(\omega),R}(\theta) = \pi\lambda_{V,\mu}(K) \int_{S^1} \phi(\theta)\, d\mathfrak{m}_1(\theta) \tag{7}$$

for any Riemann integrable function ϕ on S^1. Let $\mathcal{K}$ denote the collection of intervals $(r_1, r_2]$ such that r_1, r_2 are rational numbers and $0 < r_1 < r_2$. By the above there exists a Borel set $U \subset \mathcal{M}$, $\mu(U) = \mu(\mathcal{M})$, such that for any $\omega \in U$, any $K \in \mathcal{K}$, and any function $\phi \in C(S^1)$ relation (7) holds.

Take any $\omega \in U$. For each function $f \in L^1(S^1 \times \mathbb{R}^+, \mathfrak{m}_1 \times \lambda_{V,\mu})$ set

$$F_\omega(f) = \limsup_{R\to\infty} \Bigl| \frac{1}{\pi R^2} \int_{S^1\times\mathbb{R}^+} f(\Theta)\, dD_{V(\omega),R}(\Theta) - \int_{S^1}\int_0^\infty f(\theta,t)\, d\mathfrak{m}_1(\theta)\, d\lambda_{V,\mu}(t) \Bigr|.$$

Suppose $f \in C_c(S^1\times\mathbb{R}^+)$, i.e., f is a continuous function on $S^1\times\mathbb{R}^+$ vanishing outside $S^1 \times [1/a, a]$ for some $a > 1$. For any $\epsilon > 0$ there exist disjoint sets $K_1, \dots, K_n \in \mathcal{K}$ and functions $\phi_1, \dots, \phi_n \in C(S^1)$ such that the function f_ϵ defined by $f_\epsilon(\theta,t) = \phi_1(\theta)\chi_{K_1}(t) + \dots + \phi_n(\theta)\chi_{K_n}(t)$, $\theta \in S^1$, $t \in \mathbb{R}^+$, satisfies the inequality $\sup|f - f_\epsilon| < \epsilon$. Observe that

$$\int_{S^1\times\mathbb{R}^+} f_\epsilon(\Theta)\, dD_{V(\omega),R}(\Theta) = \sum_{i=1}^n \int_{S^1} \phi_i(\theta)\, d\delta_{W_{K_i}V(\omega),R}(\theta),$$

$$\int_{S^1}\int_0^\infty f_\epsilon(\theta,t)\, d\mathfrak{m}_1(\theta)\, d\lambda_{V,\mu}(t) = \sum_{i=1}^n \lambda_{V,\mu}(K_i) \int_{S^1} \phi_i(\theta)\, d\mathfrak{m}_1(\theta).$$

Since $\omega \in U$, it follows that $F_\omega(f_\epsilon) = 0$. Further,

$$\int_{S^1\times\mathbb{R}^+} |f(\Theta) - f_\epsilon(\Theta)|\, dD_{V(\omega),R}(\Theta) \le \epsilon N_{W_{[1/a,a]}V(\omega)}(R),$$

$$\int_{S^1}\int_0^\infty |f(\theta,t) - f_\epsilon(\theta,t)|\, d\mathfrak{m}_1(\theta)\, d\lambda_{V,\mu}(t) \le \epsilon\lambda_{V,\mu}([a^{-1},a]).$$

It follows that $F_\omega(f) \le \epsilon M(a,\omega)$, where $M(a,\omega) = \sup_{R>0} R^{-2}N_{W_{[1/a,a]}V(\omega)}(R) + \lambda_{V,\mu}([1/a,a])$. Since $M(a,\omega) < \infty$ while ϵ can be chosen arbitrarily small, we have $F_\omega(f) = 0$, i.e., equality (6) holds.

The remaining part of the proof is completely analogous to the corresponding part of the proof of Proposition 4.4 and we omit it. ∎

5 Translation surfaces. The Delaunay partitions

Let M be a compact connected oriented surface. A *translation structure* on M is an atlas of coordinate charts $\omega = \{(U_\alpha, f_\alpha)\}_{\alpha\in\mathcal{A}}$, where U_α is a domain in M and f_α is an orientation-preserving homeomorphism of U_α onto a domain in $\mathbb{R}^2$, such that:

(i) all transition functions are translations in $\mathbb{R}^2$;

(ii) chart domains U_α, $\alpha \in \mathcal{A}$, cover all surface M except for finitely many points (called *singular* points);

(iii) the atlas ω is maximal relative to the conditions (i) and (ii);

(iv) a punctured neighborhood of any singular point covers a punctured neighborhood of a point in $\mathbb{R}^2$ via an m-to-1 map which is a translation in coordinates of the atlas ω; the number m is called the *multiplicity* of the singular point.

A *translation surface* is a compact connected oriented surface equipped with a translation structure. In what follows we assume that each translation surface has at least one singular point.

Let M be a translation surface and ω be the translation structure of M. Each translation of the plane $\mathbb{R}^2$ is a smooth map preserving Euclidean metric and Lebesgue measure on $\mathbb{R}^2$. Hence the translation structure ω induces a smooth structure, a flat Riemannian metric, and a finite Borel measure on the surface M punctured at the singular points of ω. Each singular point of ω is a cone type singularity of the metric. The cone angle is equal to $2\pi m$, where m is the multiplicity of the singular point. A singular point of multiplicity 1 is called *removable* as the flat metric can be continuously extended to this point. Any geodesic of the metric is a straight line in coordinates of the atlas ω. The translation structure ω allows us to identify the tangent space at any nonsingular point $x \in M$ with the Euclidean space $\mathbb{R}^2$ so that velocity be an integral of the geodesic flow with respect to this identification. Thus each oriented geodesic is assigned a *direction* $v \in S^1$.

Let M be a compact connected oriented surface. Suppose M is endowed with a complex structure given by an atlas of charts (U_α, z_α), $\alpha \in \mathcal{A}$, and let q be a nonzero Abelian differential (holomorphic 1-form) on M. A chart (U, z) is called a *natural* *parameter* of q if $q = dz$ in U. Let ω denote the atlas of all natural parameters of q. By identifying $\mathbb{C}$ with $\mathbb{R}^2$ we can assume that charts of ω range in $\mathbb{R}^2$. Then ω becomes a translation structure on M. The singular points of ω are the zeroes of the differential q, namely, a zero of order n is a singular point of multiplicity $n+1$. Every translation structure on M without removable singular points can be obtained this way.

A simple and effective way to construct translation surfaces is to glue them from polygons. Let $P_1, \dots, P_n$ be disjoint plane polygons. The natural orientation of $\mathbb{R}^2$ induces an orientation of the boundary of every polygon. Suppose all sides of the polygons $P_1, \dots, P_n$ are distributed into pairs such that two sides in each pair are of the same length and direction, and of opposite orientations. Glue the sides in each pair by translation. Then the union of the polygons $P_1, \dots, P_n$ becomes a surface M. By construction, the surface M is compact and oriented. Suppose M is connected (if it is not, then we should apply the construction to a smaller set of polygons). The restrictions of the identity map on $\mathbb{R}^2$ to the interiors of the polygons $P_1, \dots, P_n$ can be regarded as charts of M. This finite collection of charts extends to a translation structure ω on M. The translation structure ω is uniquely determined if we require that the set of singular points of ω be the set of points corresponding to vertices of the polygons $P_1, \dots, P_n$.

Let M be a translation surface. A *saddle connection* of M is a geodesic segment joining two singular points or a singular point to itself and having no singular points in its interior. Two saddle connections of M are said to be *disjoint* if they have no common interior points (common endpoints are allowed). A domain $U \subset M$ containing no singular points is called a *triangle* (a *polygon*, an *n-gon*) if there is a map $f : U \to \mathbb{R}^2$ such that the chart (U, f) belongs to the translation structure of M and $f(U)$ is the interior of a triangle (resp. a polygon, an n-gon) in the plane $\mathbb{R}^2$. Suppose we have a finite collection of pairwise disjoint saddle connections dividing the surface M into finitely many domains such that each domain is a polygon and, moreover, each n-gon is bounded by n saddle connections (in general, an n-gon on M may be bounded by more than n saddle connections). Then M can be obtained by gluing together plane polygons as described above so that the saddle connections correspond to sides of glued polygons. The following well-known proposition shows, in particular, that any translation surface can be obtained this way.

Proposition 5.1 *(a) Any collection of pairwise disjoint saddle connections of a translation surface M can be extended to a maximal collection.*

(b) Any maximal collection of pairwise disjoint saddle connections partitions the surface M into triangles, each triangle being bounded by three saddle connections.

(c) For any maximal collection, the number of saddle connections is equal to $3m$ and the number of triangles in the corresponding triangulation is equal to $2m$, where m is the sum of multiplicities of singular points.

(d) $m = 2p - 2 + k$, where p is the genus of M and k is the number of singular points of M.

Every translation surface has a useful Delaunay partition. We shall define this partition following the paper of Masur and Smillie [MS].

Let Z be the set of singular points of a translation surface M. For any $x \in M \setminus Z$ let $d(x, Z)$ denote the distance from x to the set Z. A geodesic segment joining x to a singular point is called a *length-minimizing path* if its length is equal to $d(x, Z)$. Any point $x \in M \setminus Z$ admits at least one length-minimizing path and the number of all such paths is at most finite (see [MS]).

For any $R > 0$ let $B(R)$ denote the disk of radius R in $\mathbb{R}^2$ centered at the origin. Given a point $x \in M \setminus Z$, there is a unique map $\imath_x : B(d(x, Z)) \to M \setminus Z$ such that $\imath_x(0) = x$ and $\imath_x$ is a translation with respect to the translation structure of M. The map $\imath_x$ is uniquely extended to a continuous map of the closure of $B(d(x, Z))$ to M. Let Z'_x be the set of points in $\partial B(d(x, Z))$ that map to points in Z. Z'_x is a nonempty finite set. Each segment joining the origin to a point in Z'_x is mapped by $\imath_x$ to a length-minimizing path. The cardinality of Z'_x is equal to the number of length-minimizing paths starting at x. Let H_x be the convex hull of Z'_x. If Z'_x consists of two points then H_x is a segment and its image under $\imath_x$ is a saddle connection called a *Delaunay edge*. If Z'_x consists of more than two points then H_x is a polygon inscribed in the circle $\partial B(d(x, Z))$. It is shown in [MS] that the restriction of the map $\imath_x$ to the interior of H_x is injective. The image under $\imath_x$ of the interior of H_x is a polygon on M called a *Delaunay cell*.

Proposition 5.2 ([MS]) *Distinct Delaunay edges are disjoint saddle connections. Distinct Delaunay cells are disjoint domains. Any Delaunay cell is bounded by Delaunay edges. Any Delaunay edge separates two Delaunay cells or a Delaunay cell from itself.*

Proposition 5.2 implies that Delaunay edges partition the surface M into Delaunay cells. This partition is called the *Delaunay partition* of the translation surface M. By Proposition 5.1, the number of Delaunay edges and Delaunay cells is finite.

Proposition 5.3 *Suppose τ is a triangulation of a translation surface M by a maximal set of disjoint saddle connections. Then τ is the Delaunay partition of M if and only if for any edge L of τ two angles θ_1, θ_2 opposite L in two triangles of τ bounded by L satisfy the inequality $\theta_1 + \theta_2 < \pi$.*

Proof. By Proposition 5.2, the triangulation τ is the Delaunay partition of the translation surface M if and only if each triangle of τ is a Delaunay cell.

Denote by ω the translation structure of M. For any triangle T of τ there is a map $f_T : T \to \mathbb{R}^2$ such that $(T, f_T) \in \omega$. The map f_T is determined up to translation. We can assume without loss of generality that the triangle $f_T(T) \subset \mathbb{R}^2$ is inscribed in a circle centered at the origin. By R_T denote the radius of the circle.

Let L be an edge of τ and T_1, T_2 be the triangles of τ bounded by L. Let θ_1 and θ_2 be the angles of T_1 and T_2 opposite L. By U denote the union of T_1, T_2, and the interior of the edge L. Then U is a polygon. Let $f : U \to \mathbb{R}^2$ be a map such that $(U, f) \in \omega$ and $f = f_{T_1}$ on T_1. Suppose T_1 is a Delaunay cell. By Z denote the set of singular points of M. Then there exists a point $x \in M \setminus Z$ such that $d(x, Z) = R_{T_1}$

and $f(\imath_x(x')) = x'$ for any $x' \in f(T_1)$. It is easy to see that $f(\imath_x(x')) = x'$ for any $x' \in f(U) \cap B(R_{T_1})$. The only points in the closure of $B(R_{T_1})$ mapped by $\imath_x$ to singular points are vertices of the triangle $f(T_1)$. Therefore the vertex of the triangle $f(T_2)$ opposite its side $f(L)$ lies outside the circle $\partial B(R_{T_1})$. The latter condition is equivalent to the inequality $\theta_1 + \theta_2 < \pi$.

Now suppose that for any edge L of τ the sum of two angles opposite L is less than π. Let T be a triangle of τ, $x_0 \in T$, and I be a geodesic segment joining x_0 to a singular point. Further, let $x'_0 = f_T(x_0)$ and y' be a point in $\mathbb{R}^2$ such that the vector $y' - x'_0$ is of the same length and direction as I. We claim that y' lies outside the circle $\partial B(R_T)$ unless it is a vertex of the triangle $f_T(T)$. It easily follows from the claim that T is a Delaunay cell. The claim is proved by induction on the number n of times the segment I intersects edges of the triangulation τ. In the case $n = 0$ the segment I does not leave the triangle T hence the point y' is a vertex of $f_T(T)$. Now let $n > 0$ and assume the claim holds for all smaller numbers of intersections. Let L be the edge of τ first intersected by I and T_1 be the triangle of τ separated from T by L. Let U denote the union of triangles T, T_1, and the interior of the edge L. There exists a map $f : U \to \mathbb{R}^2$ such that $(U, f) \in \omega$ and $f = f_T$ on T. Let z' be the center of the circle circumscribed around the triangle $f(T_1)$. Then $f(x) = z' + f_{T_1}(x)$ for all $x \in T_1$. Take a point $x_1 \in T_1 \cap I$ such that the subsegment of I joining x_0 to x_1 intersects L only once and does not intersect the other edges of τ. By I_1 denote the subsegment of I joining x_1 to a singular point. Then the point $x'_1 = f(x_1)$ lies on the segment joining x'_0 to y' and the vector $y' - x'_1$ is of the same length and direction as I_1. By construction, I_1 has $n - 1$ intersections with edges of τ. By the inductive assumption the point y' lies outside the circle $z' + \partial B(R_{T_1})$ circumscribed around $f(T_1)$ unless y' is the vertex of $f(T_1)$ opposite the side $f(L)$. Since the sum of two angles opposite $f(L)$ in triangles $f(T)$ and $f(T_1)$ is less than π, it follows that the disk $z' + B(R_{T_1})$ contains the part of the disk $B(R_T)$ separated from the triangle $f(T)$ by $f(L)$. Hence y' lies outside the circle $\partial B(R_T)$. The claim is proved and so is the proposition. ∎

Lemma 5.4 *Suppose M is a translation surface of area at most 1, $x \in M$, and d is the distance from x to the closest singular point of M. If $d \geq \sqrt{2/\pi}$ then x belongs to a periodic cylinder of length at most d^{-1}.*

Proof. There is a map $\imath : B(d) \to M$ such that $\imath(0) = x$ and $\imath$ is a translation with respect to the translation structure of M. Let $r > 0$ be the maximal number such that $\imath$ is injective on $B(r)$. Then $\pi r^2 \leq 1$ since the area of M is at most 1. Assuming $d \geq \sqrt{2/\pi}$, one has $r < d$. By the choice of r, there are distinct points $x'_1, x'_2 \in \partial B(r)$ such that $\imath(x'_1) = \imath(x'_2)$. It is easy to observe that the segment joining x'_1 to x'_2 is a diameter of the disk $B(r)$ and $\imath$ maps the segment to a periodic geodesic passing through x. Hence x belongs to a periodic cylinder C of length $2r$. Let w denote the width of C. The area of the cylinder is equal to $2rw$, therefore $2rw \leq 1$. The Pythagorean theorem implies the distance from x to the closest singular point on the boundary of C is at most $\sqrt{(w/2)^2 + r^2}$. Hence $d^2 \leq (w/2)^2 + r^2$. Since $d \geq \sqrt{2/\pi}$, we have $r^2 \leq d^2/2$, then $d^2/2 \leq (w/2)^2$. In particular, $2r \leq w^{-1} \leq d^{-1}$. ∎

6 Moduli spaces of translation surfaces

Let M, M' be translation surfaces and ω, ω' be their translation structures. An orientation-preserving homeomorphism $f : M \to M'$ is called an *isomorphism* of the translation surfaces if f maps the set of singular points of M onto the set of singular points of M' and f is a translation in local coordinates of the atlases ω and ω'. The translation structures ω and ω' are called *isomorphic* if there is an isomorphism $f : M \to M'$. If $M = M'$ and $\omega = \omega'$ then the isomorphism f is called an *automorphism* of ω. Automorphisms of the translation structure ω form a group $\mathrm{Aut}(\omega)$, which is finite.

Given positive integers p and n, let M_p be a compact connected oriented surface of genus p and Z_n be a subset of M_p of cardinality n. Denote by $\Omega(p,n)$ the set of translation structures on M_p such that Z_n is the set of singular points. Let $\omega = \{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ be a translation structure on M_p. Given an orientation-preserving homeomorphism $f : M_p \to M_p$, the atlas $\omega f = \{(f^{-1}(U_\alpha), \phi_\alpha f)\}_{\alpha\in\mathcal{A}}$ is a translation structure on M_p isomorphic to ω. Let $H(p,n)$ denote the group of orientation-preserving homeomorphisms of the surface M_p leaving invariant the set Z_n. By $H_0(p,n)$ denote the subgroup of $H(p,n)$ consisting of homeomorphisms isotopic to the identity. For any $\omega \in \Omega(p,n)$ and $f \in H(p,n)$ the translation structure ωf belongs to $\Omega(p,n)$. Isomorphic translation structures $\omega, \omega' \in \Omega(p,n)$ are called *isotopic* if $\omega' = \omega f_0$ for some $f_0 \in H_0(p,n)$. Let $\mathcal{Q}(p,n)$ denote the set of equivalence classes of isotopic translation structures in $\Omega(p,n)$ and $\mathcal{MQ}(p,n)$ denote the set of equivalence classes of isomorphic translation structures in $\Omega(p,n)$. The map $H(p,n) \times \Omega(p,n) \ni (f,\omega) \mapsto \omega f^{-1}$ defines an action of the group $H(p,n)$ on the set $\Omega(p,n)$. By definition, $\mathcal{Q}(p,n) = \Omega(p,n)/H_0(p,n)$ and $\mathcal{MQ}(p,n) = \Omega(p,n)/H(p,n)$. The mapping class group $\mathrm{Mod}(p,n) = H(p,n)/H_0(p,n)$ acts naturally on the set $\mathcal{Q}(p,n)$ and $\mathcal{MQ}(p,n) = \mathcal{Q}(p,n)/\operatorname{Mod}(p,n)$.

Any translation structure on a surface of genus p with n singular points is isomorphic to a translation structure in $\Omega(p,n)$. Thus $\mathcal{MQ}(p,n)$ is the *moduli space* of translation surfaces of genus p with n singular points.

As mentioned in Section 5, there is a one-to-one correspondence between translation structures in $\Omega(p,n)$ and pairs (X,q) such that X is a complex structure on M_p and q is an Abelian differential of X whose zeroes are contained in Z_n. This allows one to regard $\mathcal{MQ}(p,n)$ as the moduli space of Abelian differentials on Riemann surfaces of genus p with at most n zeroes.

The set $\mathcal{Q}(p,n)$ has the natural structure of an affine manifold while the moduli space $\mathcal{MQ}(p,n)$ has the structure of an affine orbifold. We shall describe these structures following the paper [MS].

Let $\gamma : [0,1] \to M_p$ be a continuous path. For any $\omega \in \Omega(p,n)$ there exists a continuous path $\gamma_\omega : [0,1] \to \mathbb{R}^2$ such that γ is a translation of γ_ω in coordinates of the atlas ω. The path γ_ω is determined up to translation. The vector $\gamma_\omega(1) - \gamma_\omega(0)$ is called the *holonomy vector* of γ with respect to ω and is denoted by $\mathrm{hol}_\omega(\gamma)$. If γ is a geodesic segment of ω, then the vector $\mathrm{hol}_\omega(\gamma)$ is of the same length and direction as γ. Suppose q is an Abelian differential associated to the translation structure ω (see

Section 5). If the path γ is piecewise smooth then, up to the natural identification of $\mathbb{C}$ with $\mathbb{R}^2$,

$$\mathrm{hol}_\omega(\gamma) = \int_\gamma q.$$

The holonomy vector $\mathrm{hol}_\omega(\gamma)$ does not change when we replace the path γ by a homologous one. If the path γ is closed or its endpoints belong to the set Z_n, then $\mathrm{hol}_\omega(\gamma)$ does not change when we replace the translation structure ω by an isotopic one. The map $\gamma \mapsto \mathrm{hol}_\omega(\gamma)$ gives rise to a map $\mathrm{dev}(\omega) : H_1(M_p, Z_n; \mathbb{Z}) \to \mathbb{R}^2$ that is an element of the relative cohomology group $H^1(M_p, Z_n; \mathbb{R}^2)$. As the relative cohomology class $\mathrm{dev}(\omega)$ depends only on the isotopy class of the translation structure ω, we have a well-defined map $\mathrm{dev} : \mathcal{Q}(p, n) \to H^1(M_p, Z_n; \mathbb{R}^2)$. The group $H^1(M_p, Z_n; \mathbb{R}^2)$ is a real vector space of dimension $2N$, where $N = 2p + n - 1$ is the rank of the relative homology group $H_1(M_p, Z_n; \mathbb{Z})$.

Let $T(p, n)$ denote the set of pairs (ω, τ), where $\omega \in \Omega(p, n)$ and τ is a partition of the surface M_p by a maximal set of pairwise disjoint saddle connections of the translation structure ω. By Proposition 5.1, cells of the partition τ are triangles with respect to ω. We call the pair (ω, τ) a *triangulation*. A homeomorphism $f \in H(p, n)$ maps τ to a partition $f\tau$, which is a partition by disjoint saddle connections of the translation structure ωf^{-1}. The map $H(p, n) \times T(p, n) \ni (f, (\omega, \tau)) \mapsto (\omega f^{-1}, f\tau)$ defines an action of the group $H(p, n)$ on $T(p, n)$. Two elements $(\omega_1, \tau_1), (\omega_2, \tau_2) \in T(p, n)$ are called *affine equivalent* if there exists $f_0 \in H_0(p, n)$ such that f_0 maps each triangle of τ_1 to a triangle of τ_2 and f_0 is an affine map in local coordinates of the atlases ω_1 and ω_2 when restricted to any triangle of τ_1. By $\mathcal{T}(p, n)$ denote the set of affine equivalence classes of the above triangulations. $\mathcal{T}(p, n)$ is a countable set. The affine equivalence relation is preserved by the action of $H(p, n)$ and, moreover, each equivalence class is invariant under the action of the subgroup $H_0(p, n)$. Hence the $H(p, n)$ action on $T(p, n)$ gives rise to an action of the group $\mathrm{Mod}(p, n)$ on $\mathcal{T}(p, n)$. Let $\mathcal{MT}(p, n) = \mathcal{T}(p, n)/\operatorname{Mod}(p, n)$. An element of $\mathcal{MT}(p, n)$ can be regarded as a pattern to glue a translation surface of genus p with n singular points from triangles. Since the number of triangles to be glued together is fixed, the set $\mathcal{MT}(p, n)$ is finite.

For any $\tau \in \mathcal{T}(p, n)$ let $N(\tau)$ denote the set of translation structures in $\Omega(p, n)$ that admit a triangulation in the class τ. The set $N(\tau)$ is invariant under the action of the group $H_0(p, n)$, therefore we shall consider $N(\tau)$ as a subset of $\mathcal{Q}(p, n)$.

Lemma 6.1 *For any $\tau \in \mathcal{T}(p, n)$ the restriction of the map* dev *to the set $N(\tau)$ is injective. The image* $\mathrm{dev}(N(\tau))$ *is an open set that can be determined by a system of inequalities $Q_i(v) > 0$, $1 \le i \le s$, where $Q_1, \dots, Q_s$ are quadratic forms on the vector space $H^1(M_p, Z_n; \mathbb{R}^2)$. For any $\tau_1, \tau_2 \in \mathcal{T}(p, n)$ the set* $\mathrm{dev}(N(\tau_1) \cap N(\tau_2))$ *is also open.*

Proof. Let $(\omega_0', \tau_0) \in T(p, n)$, $\tau \in \mathcal{T}(p, n)$ be the affine equivalence class of (ω_0', τ_0), and $\omega_0 \in \mathcal{Q}(p, n)$ be the isotopy class of ω_0'. Take any triangle T of the triangulation τ_0. There is a map $f_T : T \to \mathbb{R}^2$ such that $(T, f_T) \in \omega_0'$. Let Λ denote the euclidean area form on $\mathbb{R}^2$. For any vectors $v = (v_1, v_2)$ and $u = (u_1, u_2)$, $\Lambda(v, u) = v_1 u_2 - u_1 v_2$.

Let γ_{1T} and γ_{2T} be two sides of T oriented so that $\Lambda(\mathrm{hol}_{\omega_0}(\gamma_{1T}), \mathrm{hol}_{\omega_0}(\gamma_{2T})) > 0$. For any $q \in H^1(M_p, Z_n; \mathbb{R}^2)$ let $Q_T(q) = \Lambda(q(\gamma_{1T}), q(\gamma_{2T}))$. Clearly, Q_T is a quadratic form on $H^1(M_p, Z_n; \mathbb{R}^2)$. Note that $Q_T(\mathrm{dev}(\omega_0))$ is the area of the triangle T. We claim that $\mathrm{dev}(N(\tau))$ is the set of $q \in H^1(M_p, Z_n; \mathbb{R}^2)$ such that $Q_T(q) > 0$ for any triangle T of τ_0. Given $\omega \in N(\tau)$, for any triangle T of τ_0 there exists a linear map $b_T : \mathbb{R}^2 \to \mathbb{R}^2$ such that charts $(T, b_T f_T)$ belong to a translation structure $\omega' \in \omega$. Clearly, each operator b_T is invertible and orientation-preserving. Then

$$Q_T(\mathrm{dev}(\omega)) = \Lambda(b_T \,\mathrm{hol}_{\omega_0}(\gamma_{1T}), b_T \,\mathrm{hol}_{\omega_0}(\gamma_{2T})) = (\det b_T)\, Q_T(\mathrm{dev}(\omega_0)) > 0$$

for all T. If $\mathrm{dev}(\omega) = \mathrm{dev}(\omega_0)$ then each b_T is the identity. Since charts (T, f_T) uniquely determine the translation structure ω'_0, it follows that $\omega = \omega_0$. Thus the map dev is injective on $N(\tau)$. Now consider any $q \in H^1(M_p, Z_n; \mathbb{R}^2)$ such that $Q_T(q) > 0$ for all triangles T of τ_0. For each T let b_T be a linear operator in $\mathbb{R}^2$ such that $b_T \,\mathrm{hol}_{\omega_0}(\gamma_{1T}) = q(\gamma_{1T})$, $b_T \,\mathrm{hol}_{\omega_0}(\gamma_{2T}) = q(\gamma_{2T})$. Since $Q_T(q) > 0$, b_T is invertible and orientation-preserving. If γ is a common edge of triangles T_1 and T_2, then $b_{T_1} \mathrm{hol}_{\omega_0}(\gamma) = b_{T_2} \mathrm{hol}_{\omega_0}(\gamma) = q(\gamma)$. Hence the set of charts $(T, b_T f_T)$ extends to a translation structure $\omega'_1 \in \Omega(p, n)$. By construction $(\omega'_1, \tau_0) \in \tau$, therefore the isotopy class ω_1 of ω'_1 belongs to $N(\tau)$. Also, $\mathrm{dev}(\omega_1) = q$ since the two cohomologies take the same values on edges of τ_0.

Suppose $\omega_0 \in N(\tau_1) \cap N(\tau_2)$ for some $\tau_1, \tau_2 \in \mathcal{T}(p, n)$. Let ω'_0 be a translation structure in the isotopy class ω_0 and τ'_1, τ'_2 be triangulations such that $(\omega'_0, \tau'_1) \in \tau_1$, $(\omega'_0, \tau'_2) \in \tau_2$. Choose an edge γ of τ'_2. If γ is not an edge of τ'_1 then edges of τ'_1 break this saddle connection into several geodesic segments $x_0x_1, x_1x_2, \dots, x_kx_{k+1}$, where x_0 and x_{k+1} are endpoints of γ and $x_1, \dots, x_k$ are interior points of edges of τ'_1. Take any $\epsilon > 0$. Let $x'_1, \dots, x'_k$ be points of M_p such that each x'_i $(1 \le i \le k)$ lies on the same edge of τ'_1 as x_i and the length of the subsegment of the edge bounded by x'_i and x_i is less than ϵ. Assuming ϵ is small enough, there are geodesic segments $x_0x'_1, x'_1x'_2, \dots, x'_kx_{k+1}$ that form a path homotopic to γ. By $U_\epsilon(\gamma)$ denote the set of all such piecewise geodesics. If γ is a common edge of τ'_1 and τ'_2, let $U_\epsilon(\gamma) = \{\gamma\}$. Let $\gamma_1, \dots, \gamma_l$ be all edges of τ'_2. If ϵ is small enough then for any $\gamma'_j \in U_\epsilon(\gamma_j)$, $1 \le j \le l$, the curves $\gamma'_1, \dots, \gamma'_l$ are simple and disjoint except for endpoints. The partition of M_p by these curves can be mapped onto τ'_2 by a homeomorphism $f \in H_0(p, n)$. By P_ϵ denote the set of all such partitions. For any $\omega \in N(\tau_1)$ there exists $\omega' \in \omega$ such that $(\omega', \tau'_1) \in \tau_1$ and the identity map of M_p is affine on every triangle of τ'_1 with respect to translation structures ω'_0 and ω'. If $\mathrm{dev}(\omega)$ is close to $\mathrm{dev}(\omega_0)$ then linear parts of the restrictions to triangles of τ'_1 are close to 1. It follows that if $\mathrm{dev}(\omega)$ is close enough to $\mathrm{dev}(\omega_0)$ then each of the sets $U_\epsilon(\gamma_1), \dots, U_\epsilon(\gamma_l)$ contains a saddle connection of the translation structure ω'. Then P_ϵ contains a partition τ' such that $(\omega', \tau') \in \tau_2$, hence $\omega \in N(\tau_2)$. Thus $\mathrm{dev}(\omega_0)$ is an interior point of $\mathrm{dev}(N(\tau_1) \cap N(\tau_2))$. ∎

By Proposition 5.1, the sets $N(\tau)$, $\tau \in \mathcal{T}(p, n)$, cover $\mathcal{Q}(p, n)$. We use this covering and the map dev to endow $\mathcal{Q}(p, n)$ with a topology, smooth and affine structures, and a measure. By Lemma 6.1, there exists a unique topology on $\mathcal{Q}(p, n)$ such that any $N(\tau)$ is an open set and the restriction of dev to $N(\tau)$ is a homeomorphism onto $\text{dev}(N(\tau))$. The topological space $\mathcal{Q}(p,n)$ is a $2N$-dimensional manifold. Further, there is a unique affine (smooth, real analytic) structure on $\mathcal{Q}(p,n)$ such that the restriction of the map dev to any $N(\tau)$, $\tau \in \mathcal{T}(p,n)$, is an affine (resp. smooth, analytic) map. The euclidean volume form on the vector space $H^1(M_p, Z_n; \mathbb{R}^2)$ is made canonical by the requirement that the lattice $H^1(M_p, Z_n; \mathbb{Z}^2)$ have covolume 1. The volume element on $\mathcal{Q}(p,n)$ is obtained from the pull-back, this volume form by the map dev.

Lemma 6.2 *Let $\omega \in \Omega(p,n)$. Then each set of disjoint saddle connections of ω that do not divide the surface M_p can be extended to a maximal set of this kind. Each maximal set consists of $N = 2p + n - 1$ saddle connections whose homology classes comprise a basis for $H_1(M_p, Z_n; \mathbb{Z})$.*

Proof. Let $\gamma_1, \dots, \gamma_k$ be disjoint saddle connections of ω that do not divide the surface M_p. Let $m = 2p - 2 + n$. By Proposition 5.1, $k \le 3m$ and there exist $\gamma_{k+1}, \dots, \gamma_{3m}$ such that $\gamma_1, \dots, \gamma_{3m}$ are disjoint saddle connections of ω that form a triangulation τ of M_p. There are $2m$ triangles in the triangulation τ. We can arrange them in a sequence $T_0, T_1, \dots, T_{2m-1}$ so that there exist a sequence of domains $U_0 = T_0 \subset U_1 \subset \ldots \subset U_{2m-1} \subset M_p$ and a sequence of edges $e_1, \dots, e_{2m-1}$ of τ such that for any j, $1 \le j \le 2m - 1$, the triangle T_j is disjoint from U_{j-1}, e_j separates T_j from U_{j-1}, and U_j is the union of U_{j-1}, T_j, and the interior of e_j. The complement of U_{2m-1} is the union of the set of singular points of ω and $N = m + 1$ edges of τ. Since saddle connections $\gamma_1, \dots, \gamma_k$ do not divide the surface M_p, it can be assumed without loss of generality that all of them are in the complement of U_{2m-1}.

It is easy to observe that the group $H_1(M_p, Z_n; \mathbb{Z})$ is generated by the homology classes of edges of τ. For any j, $1 \le j \le 2m - 1$, two sides of the triangle T_j different from e_j are disjoint from U_j. It follows that $H_1(M_p, Z_n; \mathbb{Z})$ is generated by the homology classes of N edges of τ disjoint from U_{2m-1}. Since $H_1(M_p, Z_n; \mathbb{Z})$ is a free Abelian group of rank N, the latter classes constitute its basis. ∎

Suppose $\Gamma = (\gamma_1, \dots, \gamma_N)$ is an ordered basis of cycles for $H_1(M_p, Z_n; \mathbb{Z})$. Define a map $f_\Gamma : H^1(M_p, Z_n; \mathbb{R}^2) \to (\mathbb{R}^2)^N \approx \mathbb{R}^{2N}$ by $f_\Gamma(q) = (q(\gamma_1), \dots, q(\gamma_N))$. f_Γ is an isomorphism of vector spaces. Since $f_\Gamma(H^1(M_p, Z_n; \mathbb{Z}^2)) = \mathbb{Z}^{2N}$, f_Γ preserves volumes. Now define a map $F_\Gamma : \mathcal{Q}(p,n) \to (\mathbb{R}^2)^N \approx \mathbb{R}^{2N}$ by $F_\Gamma(\omega) = (\text{hol}_\omega(\gamma_1), \dots, \text{hol}_\omega(\gamma_N))$. Clearly, $F_\Gamma(\omega) = f_\Gamma(\text{dev}(\omega))$. The map F_Γ is a local homeomorphism. The volume element on $\mathcal{Q}(p,n)$ is the pull-back of the canonical volume form on $\mathbb{R}^{2N}$ by the map F_Γ.

Lemma 6.3 *The group* $\text{Mod}(p,n)$ *acts on* $\mathcal{Q}(p,n)$ *by affine homeomorphisms preserving volume element. The action is properly discontinuous.*

Proof. Every mapping class $\phi \in \text{Mod}(p,n)$ induces an automorphism ϕ_* of the group $H_1(M_p, Z_n; \mathbb{Z})$. Define a linear operator ϕ^* on $H^1(M_p, Z_n; \mathbb{R}^2)$ by $(\phi^* q)(\gamma) = q(\phi_* \gamma)$, $\gamma \in H_1(M_p, Z_n; \mathbb{Z})$. ϕ^* is invertible and preserves the lattice $H^1(M_p, Z_n; \mathbb{Z}^2)$, hence it is volume preserving. Observe that $\text{dev}(\omega\phi^{-1}) = \phi^*(\text{dev}(\omega))$ for all $\omega \in \mathcal{Q}(p,n)$. Besides, if $\omega \in N(\tau)$ for some $\tau \in \mathcal{T}(p,n)$ then $\omega\phi^{-1} \in N(\phi\tau)$. It follows that the $\text{Mod}(p,n)$ action on $\mathcal{Q}(p,n)$ is affine and preserves the volume element.

Let $\omega_0 \in N(\tau)$, $\tau \in \mathcal{T}(p,n)$. Choose a triangulation $(\omega_0', \tau_0') \in \tau$ such that ω_0' is in the isotopy class ω_0. There exists a neighborhood $U \subset N(\tau)$ of ω_0 such that for every translation structure ω' in an isotopy class $\omega \in U$ any saddle connection of ω' of length l is homotopic to a piecewise geodesic path of ω_0' of length at most $2l$, while any saddle connection of ω_0' of length l_0 is homotopic to a piecewise geodesic path of ω' of length at most $2l_0$. Let us show that $U\phi^{-1} \cap U \neq \emptyset$ for only finitely many $\phi \in \text{Mod}(p,n)$. Denote by K the set of relative homotopy classes of edges of τ_0'. Let R be the maximal length of edges of τ_0'. By K_1' denote the set of paths γ on M_p with endpoints in Z_n such that γ is a piecewise geodesic path of length at most $4R$ with respect to ω_0'. For any $\gamma \in K_1'$ there is a homotopic path $\gamma' \in K_1'$ that is a saddle connection or the sum of several saddle connections of ω_0'. By K_1 denote the set of relative homotopy classes of paths in K_1'. Clearly, $K \subset K_1$. Since the translation structure ω_0' has only finitely many saddle connections of length at most $4R$, the set K_1 is finite. Suppose $U\phi^{-1} \cap U \neq \emptyset$ for some $\phi \in \text{Mod}(p,n)$, i.e., $\omega_1\phi^{-1} = \omega_2$ for some $\omega_1, \omega_2 \in U$. Pick translation structures $\omega_1' \in \omega_1$ and $\omega_2' \in \omega_2$. Let τ_1' be a partition such that $(\omega_1', \tau_1') \in \tau$. By the choice of U, all edges of τ_1' are of length at most $2R$. There is a homeomorphism $\phi_0 \in \phi$ that maps each edge of τ_1' onto a saddle connection of ω_2' of the same length. By the choice of U, the mapping class ϕ sends each homotopy class from K to a homotopy class in K_1. Since ϕ is uniquely determined by its action on elements of K and K_1 is a finite set, there are only finitely many such ϕ. Furthermore, there exists a neighborhood $U_0 \subset U$ of ω_0 such that $U_0\phi^{-1} \cap U_0 \neq \emptyset$ for a $\phi \in \text{Mod}(p,n)$ if and only if $\phi\omega_0 = \omega_0$, i.e., ϕ is the mapping class of an automorphism of ω_0'. ∎

Since the action of $\text{Mod}(p,n)$ on $\mathcal{Q}(p,n)$ is properly discontinuous and volume preserving, the quotient space $\mathcal{MQ}(p,n)$ inherits the structure of an affine orbifold along with a volume element.

Lemma 6.4 *The moduli space $\mathcal{MQ}(p,n)$ has finitely many connected components.*

Proof. It follows from Lemma 6.1 that any of the open sets $N(\tau)$, $\tau \in \mathcal{T}(p,n)$, has only finitely many connected components. Since $\mathcal{MT}(p,n)$ is a finite set, there exist $\tau_1, \dots, \tau_k \in \mathcal{T}(p,n)$ such that the natural projection $\pi_0 : \mathcal{Q}(p,n) \to \mathcal{MQ}(p,n)$ maps the union $N(\tau_1) \cup \ldots \cup N(\tau_k)$ onto $\mathcal{MQ}(p,n)$. Then the number of connected components of $\mathcal{MQ}(p,n)$ is at most the number of connected components of $N(\tau_1) \cup \ldots \cup N(\tau_k)$, which is finite. ∎

Let $\alpha = (m_1, \dots, m_n)$ be a nondecreasing sequence of positive integers such that $m_1 + \dots + m_n = 2p - 2 + n$. By $\mathcal{H}(\alpha)$ denote the set of isomorphy classes of translation surfaces with n singular points of multiplicities $m_1, \dots, m_n$. $\mathcal{H}(\alpha)$ is called a *stratum*. The stratum $\mathcal{H}(\alpha)$ is a nonempty open subset of $\mathcal{MQ}(p,n)$. The moduli space $\mathcal{MQ}(p,n)$ is a disjoint union of strata. By Lemma 6.4, each stratum has only finitely many connected components. The complete classification of all connected components for each stratum was obtained by Kontsevich and Zorich [KZ]. In particular, any stratum has at most 3 connected components.

For any $\tau \in \mathcal{T}(p,n)$ let $M(\tau)$ denote the set of translation structures $\omega \in \Omega(p,n)$ such that the Delaunay partition of ω is a triangulation in the class τ. Let $\prec$ represent an ordering of the homology classes of edges of the triangulation τ. Note that edges of τ need not be oriented so $\prec$ is actually an ordering of pairs of opposite homology classes. By $M(\tau,\prec)$ denote the set of translation structures $\omega \in M(\tau)$ such that for any two nonhomologous Delaunay edges e_1 and e_2 of ω the edge e_1 is shorter than e_2 if the homology class of e_1 is less than the homology class of e_2 in the ordering $\prec$. Let h be one of the least homology classes with respect to $\prec$ (the other is $-h$). By $M^h(\tau,\prec)$ denote the set of $\omega \in M(\tau,\prec)$ such that the holonomy vector $\mathrm{hol}_\omega(h)$ belongs to the halfplane $\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0\}$. The sets $M(\tau)$, $M(\tau,\prec)$, and $M^h(\tau,\prec)$ are invariant under the action of $H_0(p,n)$ so we consider them as subsets of $\mathcal{Q}(p,n)$. Then $M^h(\tau,\prec) \subset M(\tau,\prec) \subset M(\tau) \subset N(\tau)$. $M(\tau,\prec)$ is called a *Delaunay triangulation piece* while $M^h(\tau,\prec)$ is called a *halfpiece*. Clearly, the sets $M(\tau)$, $\tau \in \mathcal{T}(p,n)$, are disjoint. Two Delaunay triangulation pieces $M(\tau_1,\prec_1)$ and $M(\tau_2,\prec_2)$ are disjoint unless $\tau_1 = \tau_2$ and $\prec_1=\prec_2$. Two halfpieces $M^{h_1}(\tau_1,\prec_1)$ and $M^{h_2}(\tau_2,\prec_2)$ are disjoint unless $\tau_1=\tau_2$, $\prec_1=\prec_2$, and $h_1=h_2$.

Lemma 6.5 *$M(\tau)$, $M(\tau,\prec)$, and $M^h(\tau,\prec)$ are open subsets of $\mathcal{Q}(p,n)$. The natural projection $\pi_0 : \mathcal{Q}(p,n) \to \mathcal{MQ}(p,n)$ is injective on $M^h(\tau,\prec)$. There exist finitely many disjoint Delaunay triangulation halfpieces $M^{h_1}(\tau_1,\prec_1),\dots,M^{h_k}(\tau_k,\prec_k)$ such that π_0 is injective on their union and $\pi_0(M^{h_1}(\tau_1,\prec_1)\cup\ldots\cup M^{h_k}(\tau_k,\prec_k))$ is a subset of $\mathcal{MQ}(p,n)$ of full volume.*

Proof. Let $\omega' \in \Omega(p,n)$ be a representative of an $\omega \in N(\tau)$ and τ' be a triangulation of M_p such that $(\omega',\tau') \in \tau$. Let e_0 be an edge of τ' and T_1, T_2 be triangles of τ' bounded by e_0. By θ_i ($i=1,2$) denote the angle of T_i opposite e_0. Let e_1, e_2 be sides of T_1 different from e_0. Then $\cos\theta_1 = (2l_1l_2)^{-1}(l_1^2+l_2^2-l_0^2)$, where $l_i = |\,\mathrm{hol}_\omega(e_i)|$, $i=0,1,2$. The homotopy classes of e_0, e_1, and e_2 depend only on τ. It follows that $\cos\theta_1$ depends continuously on ω. Likewise, $\cos\theta_2$ is a continuous function of $\omega \in N(\tau)$. Since $0<\theta_1,\theta_2<\pi$, we have $\theta_1+\theta_2<\pi$ if and only if $\cos\theta_1+\cos\theta_2>0$. By Proposition 5.3, $M(\tau)$ can be determined as the set of $\omega \in N(\tau)$ satisfying inequalities $f_i(\omega)>0$, $i=1,\dots,k$, where $f_1,\dots,f_k$ are continuous functions on $N(\tau)$. Hence $M(\tau)$ is open. Now $M(\tau,\prec)$ is the intersection of $M(\tau)$ with a finite number of open sets of the form $\{\omega \in \mathcal{Q}(p,n) : |\,\mathrm{hol}_\omega(h_1)| < |\,\mathrm{hol}_\omega(h_2)|\}$, where $h_1,h_2 \in H_1(M_p,Z_n;\mathbb{Z})$. The halfpiece $M^h(\tau,\prec)$ is the intersection of $M(\tau,\prec)$ with the open set $\{\omega \in \mathcal{Q}(p,n) : \mathrm{hol}_\omega(h) \in \mathbb{R}^2_+\}$.

For any mapping class $\phi \in \mathrm{Mod}(p,n)$ let ϕ_* denote the induced automorphism of $H_1(M_p,Z_n;\mathbb{Z})$. ϕ acts on $\mathcal{Q}(p,n)$ so that any $M(\tau)$ is mapped onto $M(\tau)\phi^{-1} = M(\phi\tau)$. A Delaunay triangulation piece $M(\tau,\prec)$ is mapped onto $M(\phi\tau,\prec')$, where by definition $e_1 \prec' e_2$ if and only if $\phi_*^{-1}e_1 \prec \phi_*^{-1}e_2$. A halfpiece $M^h(\tau,\prec)$ is mapped onto $M^{\phi_*h}(\phi\tau,\prec')$. Suppose $M^h(\tau,\prec)\phi^{-1} = M^h(\tau,\prec)$. Then ϕ permutes homotopy classes of edges of τ. Moreover, $\phi_*h_0 = \pm h_0$ for any homology class h_0 of an edge. Since ϕ is orientation-preserving and homology classes of edges of τ generate

$H_1(M_p, Z_n; \mathbb{Z})$, it follows that ϕ_* is the multiplication by 1 or -1. As $\phi_* h = h$, ϕ_* is the identity. Let $\omega_0 \in \Omega(p, n)$ be a translation structure whose isotopy class is in $M^h(\tau, \prec)$. By the above there exists $\phi_0 \in \phi$ that maps each Delaunay cell of ω_0 onto another Delaunay cell in an affine way. ϕ_0 maps each Delaunay edge onto a homologous one, hence ϕ_0 is an automorphism of ω_0. Let e_1 and e_2 be two Delaunay edges of ω_0 bounding the same triangle. If $\phi_0(e_1) \neq e_1$ then the edges e_1 and $\phi_0(e_1)$ divide the surface M_p into two parts. It is easy to see that the edges e_2 and $\phi_0(e_2)$ are contained in different parts. None of the latter divides the part it belongs to. As a consequence, e_2 and $\phi_0(e_2)$ are not homologous as they should be. This contradiction implies ϕ_0 fixes all Delaunay edges of ω_0. Then ϕ_0 is the identity while ϕ is the trivial mapping class. It follows that the halfpiece $M^h(\tau, \prec)$ is disjoint from $M^h(\tau, \prec)\phi^{-1}$ for any nontrivial $\phi \in \mathrm{Mod}(p, n)$, i.e., π_0 is injective on $M^h(\tau, \prec)$.

Let U_1 denote the subset of $\mathcal{Q}(p, n)$ corresponding to translation structures whose Delaunay partitions are not triangulations. Let us show that U_1 has zero volume. Take any $\tau \in \mathcal{T}(p, n)$ and pick an edge e of τ (edges of τ are determined up to homotopy). By $U_1(\tau, e)$ denote the set of $\omega \in N(\tau)$ such that each Delaunay edge is an edge of τ but there is no Delaunay edge homotopic to e. U_1 is the union of countably many sets of the form $U_1(\tau, e)$, hence it is sufficient to prove that any of them has zero volume. Let ω' be a translation structure in an isotopy class $\omega \in U_1(\tau, e)$ and τ' be a partition of M_p such that $(\omega', \tau') \in \tau$. Let e_0 be an edge of τ' homotopic to e. Let T_1 and T_2 be triangles of τ' bounded by e_0. Then the union of T_1, T_2, and the interior of e_0 is isometric to an inscribed quadrilateral. Let e_1 be a side of T_1 different from e_0 and e_2 be a side of T_2 different from e_0 and not parallel to e_1. If e_0, e_1, and e_2 do not divide the surface, then their homology classes are elements of a basis for $H_1(M_p, Z_n; \mathbb{Z})$ by Lemma 6.2. Once holonomy vectors $\mathrm{hol}_\omega(e_0)$ and $\mathrm{hol}_\omega(e_1)$ are fixed, the holonomy vector $\mathrm{hol}_\omega(e_2)$ lies on a fixed circle in $\mathbb{R}^2$ passing through the origin. Hence Fubini's theorem implies $U_1(\tau, e)$ has zero volume. If e_0, e_1, and e_2 do divide the surface, it is easy to observe that e_2 is homologous to the side of T_1 different from e_0 and e_1. Then T_1 and T_2 are isometric triangles and their angles opposite e_0 are right. It follows that $\mathrm{hol}_\omega(e_1)$ lies on a circle depending on $\mathrm{hol}_\omega(e_0)$. Since e_0 and e_1 do not divide M_p, Lemma 6.2 and Fubini's theorem imply again $U_1(\tau, e)$ has zero volume.

Let U_2 denote the set of $\omega \in \mathcal{Q}(p, n)$ such that $|\mathrm{hol}_\omega(\gamma_1)| = |\mathrm{hol}_\omega(\gamma_2)|$ for some disjoint nonhomologous saddle connections γ_1 and γ_2 of a translation structure in the isotopy class ω. By Lemma 6.2, homology classes of γ_1 and γ_2 are linearly independent. By Lemma 7.1 (see Section 7 below), U_2 has zero volume. Let U_3 denote the set of $\omega \in \mathcal{Q}(p, n)$ such that $\mathrm{hol}_\omega(h)$ is horizontal for a nonzero $h \in H_1(M_p, Z_n; \mathbb{Z})$. U_3 is the union of countably many codimension 1 affine submanifolds of $\mathcal{Q}(p, n)$, hence it is of zero volume.

Since the $\mathrm{Mod}(p, n)$ action on $\mathcal{T}(p, n)$ has only finitely many orbits, there exist $\tau_1, \ldots, \tau_k \in \mathcal{T}(p, n)$ such that π_0 maps the union of $M(\tau_1), \ldots, M(\tau_k)$, and U_1 onto $\mathcal{MQ}(p, n)$. For any $\tau \in \mathcal{T}(p, n)$ the set $M(\tau)$ is the union of finitely many Delaunay triangulation pieces and a subset of U_2, while each piece $M(\tau, \prec)$ is the union of two halfpieces and a subset of U_3. Since $U_1 \cup U_2 \cup U_3$ is a set of zero volume, there exist finitely many halfpieces $P_1, \dots, P_l$ such that $\pi_0(P_1 \cup \dots \cup P_l)$ is a subset of $\mathcal{MQ}(p,n)$ of full volume. By the above any $\pi_0(P_i)$ and $\pi_0(P_j)$ either are disjoint or coincide. Therefore it is no loss to assume that sets $\pi_0(P_1), \dots, \pi_0(P_l)$ are disjoint. Then $P_1, \dots, P_l$ are also disjoint and π_0 is injective on $P_1 \cup \dots \cup P_l$. ∎

Let $\omega = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ be a translation structure on M_p. For any operator $g \in \mathrm{GL}_+(2,\mathbb{R})$ the atlas $g\omega = \{(U_\alpha, g\phi_\alpha)\}_{\alpha \in \mathcal{A}}$ is a translation structure on M_p with the same singular points of the same multiplicities as ω. Although the flat metrics on M_p induced by translation structures ω and $g\omega$ need not coincide, they share the same geodesics. If $g \in \mathrm{SL}(2,\mathbb{R})$ then ω and $g\omega$ induce the same measure on M_p. The map $\mathrm{GL}_+(2,\mathbb{R}) \times \Omega(p,n) \ni (g,\omega) \mapsto g\omega$ defines an action of the group $\mathrm{GL}_+(2,\mathbb{R})$ on the set $\Omega(p,n)$. Obviously, this action commutes with the action of $H(p,n)$. Therefore the action of $\mathrm{GL}_+(2,\mathbb{R})$ descends to actions on the spaces $\mathcal{Q}(p,n)$ and $\mathcal{MQ}(p,n)$. The action of the group $\mathrm{GL}_+(2,\mathbb{R})$ on $\mathcal{Q}(p,n)$ commutes with the action of $\mathrm{Mod}(p,n)$. Since $\mathrm{hol}_{g\omega}(\gamma) = g\,\mathrm{hol}_\omega(\gamma)$ for any $g \in \mathrm{GL}_+(2,\mathbb{R})$, $\omega \in \Omega(p,n)$, and any path γ, it follows that the $\mathrm{GL}_+(2,\mathbb{R})$ action on $\mathcal{Q}(p,n)$ is affine and continuous. Moreover, the action of the subgroup $\mathrm{SL}(2,\mathbb{R})$ preserves the volume element. The action of $\mathrm{GL}_+(2,\mathbb{R})$ on $\mathcal{MQ}(p,n)$ is also continuous and the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{MQ}(p,n)$ is also volume preserving. For any $\tau \in \mathcal{T}(p,n)$ the set $N(\tau)$ is invariant under the $\mathrm{GL}_+(2,\mathbb{R})$ action on $\mathcal{Q}(p,n)$. The $\mathrm{GL}_+(2,\mathbb{R})$ action on $\mathcal{MQ}(p,n)$ leaves invariant every stratum $\mathcal{H}(\alpha) \subset \mathcal{MQ}(p,n)$ and every connected component of $\mathcal{MQ}(p,n)$.

The group $\mathbb{R}^+$ acts naturally on translation structures by scaling distances. We define actions of $\mathbb{R}^+$ on the sets $\Omega(p,n)$, $\mathcal{Q}(p,n)$, and $\mathcal{MQ}(p,n)$ by regarding $\mathbb{R}^+$ as a subgroup of $\mathrm{GL}_+(2,\mathbb{R})$.

For any $\omega \in \Omega(p,n)$, let $a(\omega)$ denote the area of the surface M_p with respect to the measure induced by ω. Since $a(t\omega) = t^2 a(\omega)$ for any $t > 0$, every translation structure $\omega \in \Omega(p,n)$ is uniquely represented as $t\omega_1$, where $a(\omega_1) = 1$ and $t \in \mathbb{R}^+$. The area $a(\omega)$ does not change when we replace ω by an isomorphic translation structure. By $\mathcal{Q}_1(p,n)$ and $\mathcal{MQ}_1(p,n)$ denote the subsets of $\mathcal{Q}(p,n)$ and $\mathcal{MQ}(p,n)$, respectively, corresponding to translation structures ω such that $a(\omega) = 1$. Let us consider a as a function on $\mathcal{Q}(p,n)$. Then it follows from the proof of Lemma 6.1 that $a(\omega)$ is locally a quadratic form of the vector $\mathrm{dev}(\omega) \in H^1(M_p, Z_n; \mathbb{R}^2)$. Hence the set $\mathcal{Q}_1(p,n) = a^{-1}(1)$ is a real analytic submanifold of $\mathcal{Q}(p,n)$ of codimension 1. This submanifold is invariant under the actions of $\mathrm{Mod}(p,n)$ and $\mathrm{SL}(2,\mathbb{R})$. The volume element on $\mathcal{Q}(p,n)$ along with the vector field $\mathrm{grad}\, a$ induce a volume element on $\mathcal{Q}_1(p,n)$. Let $\tilde{\mu}$ and $\tilde{\mu}_0$ be the Borel measures on $\mathcal{Q}(p,n)$ and $\mathcal{Q}_1(p,n)$, respectively, induced by the volume elements. Then

$$\tilde{\mu}_0(U) = 2N \cdot \tilde{\mu}(\{t\omega \mid \omega \in U,\ 0 < t \leq 1\})$$

for any Borel set $U \subset \mathcal{Q}_1(p,n)$. The action of $\mathrm{Mod}(p,n)$ on $\mathcal{Q}_1(p,n)$ is properly discontinuous and volume preserving, therefore the quotient space $\mathcal{MQ}_1(p,n)$ inherits the structure of a real analytic orbifold along with a volume element. $\mathcal{MQ}_1(p,n)$ is a suborbifold of $\mathcal{MQ}(p,n)$ invariant under the action of $\mathrm{SL}(2,\mathbb{R})$. Each connected component of $\mathcal{MQ}_1(p,n)$ is of the form $\mathcal{C} \cap \mathcal{MQ}_1(p,n)$, where $\mathcal{C}$ is a connected component of $\mathcal{MQ}(p,n)$. By μ_0 denote the Borel measure on $\mathcal{MQ}_1(p,n)$ induced by the volume element. Let $\pi_0 : \mathcal{Q}(p,n) \to \mathcal{MQ}(p,n)$ be the natural projection. Then $\mu_0(\pi_0(U)) = \tilde{\mu}_0(U)$ for any Borel set $U \subset \mathcal{Q}_1(p,n)$ such that π_0 is injective on U. The measure μ_0 is invariant under the $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{MQ}_1(p,n)$.

Theorem 6.6 ([V2], [MS]) $\mu_0(\mathcal{MQ}_1(p,n)) < \infty$.

Theorem 6.7 ([V1]) *The* $\mathrm{SL}(2,\mathbb{R})$ *action on* $\mathcal{MQ}_1(p,n)$ *is ergodic with respect to the measure* μ_0 *on each connected component of* $\mathcal{MQ}_1(p,n)$.

Now we shall define the moduli space $\mathcal{MY}(p,n)$ of pairs (M,x) such that M is a translation surface of genus p with n singular points and $x \in M$. The maps $H(p,n) \times \Omega(p,n) \times M_p \ni (f,\omega,x) \mapsto (\omega f^{-1}, f(x))$ and $\mathrm{SL}(2,\mathbb{R}) \times \Omega(p,n) \times M_p \ni (g,\omega,x) \mapsto (g\omega, x)$ define commuting actions of the groups $H(p,n)$ and $\mathrm{SL}(2,\mathbb{R})$ on the set $\Omega(p,n) \times M_p$. Let $\mathcal{Y}(p,n) = (\Omega(p,n) \times M_p)/H_0(p,n)$ and $\mathcal{MY}(p,n) = (\Omega(p,n) \times M_p)/H(p,n)$. Let $\tilde{p}_0 : \mathcal{Y}(p,n) \to \mathcal{Q}(p,n)$ and $p_0 : \mathcal{MY}(p,n) \to \mathcal{MQ}(p,n)$ be the natural projections. The group $\mathrm{Mod}(p,n)$ acts naturally on $\mathcal{Y}(p,n)$ and $\mathcal{MY}(p,n) = \mathcal{Y}(p,n)/\mathrm{Mod}(p,n)$. The $\mathrm{SL}(2,\mathbb{R})$ action on $\Omega(p,n) \times M_p$ descends to actions on $\mathcal{Y}(p,n)$ and $\mathcal{MY}(p,n)$.

Let $\tau \in \mathcal{T}(p,n)$. Take a translation structure $\omega_0 \in \Omega(p,n)$ that admits a triangulation τ_0 in the class τ. For any $\omega \in N(\tau)$ there exists a unique translation structure $X_{\omega_0,\tau_0}(\omega)$ in the isotopy class ω such that the identity map of M_p is affine on each triangle of τ_0 in local coordinates of the atlases ω_0 and $X_{\omega_0,\tau_0}(\omega)$, that is, for any triangle T of τ_0 there are a map $f : T \to \mathbb{R}^2$ and an affine map $b : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(T,f) \in \omega_0$ and $(T,bf) \in X_{\omega_0,\tau_0}(\omega)$. For any $\omega \in N(\tau)$ and $x \in M_p$ define $Y_{\omega_0,\tau_0}(\omega,x) \in \mathcal{Y}(p,n)$ to be the $H_0(p,n)$-orbit of the pair $(X_{\omega_0,\tau_0}(\omega),x) \in \Omega(p,n) \times M_p$. Then the map $Y_{\omega_0,\tau_0} : N(\tau) \times M_p \to \mathcal{Y}(p,n)$ is injective and $Y_{\omega_0,\tau_0}(N(\tau) \times M_p) = \tilde{p}_0^{-1}(N(\tau))$. The collection of maps Y_{ω_0,τ_0}, $(\omega_0,\tau_0) \in T(p,n)$, endows $\mathcal{Y}(p,n)$ with the structure of a fiber bundle over $\mathcal{Q}(p,n)$ with the fiber M_p.

The action of the group $\mathrm{Mod}(p,n)$ on $\mathcal{Y}(p,n)$ is properly discontinuous, therefore the quotient space $\mathcal{MY}(p,n)$ inherits the quotient topology. For any $\omega \in \mathcal{MQ}(p,n)$ the following conditions are equivalent: (i) translation structures in the isomorphy class ω have no automorphisms different from the identity; (ii) for any $\tilde{\omega} \in \pi_0^{-1}(\omega)$ the restriction of the projection π_0 to some neighborhood of $\tilde{\omega}$ is a homeomorphism. Let U_0 be the set of $\omega \in \mathcal{MQ}(p,n)$ satisfying these conditions. U_0 is an open dense subset of $\mathcal{MQ}(p,n)$ of full volume. The set $p_0^{-1}(U_0) \subset \mathcal{MY}(p,n)$ is a fiber bundle over U_0 with the fiber M_p. Suppose $\omega \in \mathcal{MQ}(p,n) \setminus U_0$ and $\omega_0 \in \omega$; then $p_0^{-1}(\omega)$ is homeomorphic to $M_p/\mathrm{Aut}(\omega_0)$. Slightly abusing notation, we shall consider $\mathcal{MY}(p,n)$ as a fiber bundle over $\mathcal{MQ}(p,n)$ with the fiber M_p (even though some fibers may be not homeomorphic to M_p).

The group $\mathrm{SL}(2,\mathbb{R})$ acts on the spaces $\mathcal{Y}(p,n)$ and $\mathcal{MY}(p,n)$ by homeomorphisms. The subspaces $\mathcal{Y}_1(p,n) = \tilde{p}_0^{-1}(\mathcal{Q}_1(p,n))$ and $\mathcal{MY}_1(p,n) = p_0^{-1}(\mathcal{MQ}_1(p,n))$ are invariant under these actions. Note that each connected component of $\mathcal{MY}_1(p,n)$ is of the form $p_0^{-1}(\mathcal{C})$, where $\mathcal{C}$ is a connected component of $\mathcal{MQ}_1(p,n)$.

For any $\omega_0 \in \Omega(p,n)$, let ξ_{ω_0} be the Borel measure on M_p induced by the translation structure ω_0. Let ω be the isotopy class of ω_0. The map $h_{\omega_0} : M_p \to \tilde{p}_0^{-1}(\omega)$ defined by the relation $(\omega_0, x) \in h_{\omega_0}(x)$, $x \in M_p$, is a homeomorphism. The measure $\tilde{\rho}_\omega = \xi_{\omega_0} h_{\omega_0}^{-1}$ on the fiber $\tilde{p}_0^{-1}(\omega)$ does not depend on the choice of $\omega_0 \in \omega$. Likewise, for any $\omega \in \mathcal{MQ}(p,n)$ the measures on M_p induced by translation structures in the isomorphy class ω define a Borel measure ρ_ω on $p_0^{-1}(\omega)$ (even if the fiber $p_0^{-1}(\omega)$ is not homeomorphic to M_p). The space $\mathcal{Y}_1(p,n)$, which is a fiber bundle over $\mathcal{Q}_1(p,n)$ with the fiber M_p, carries a natural measure $\tilde{\mu}_1$ that is the measure $\tilde{\mu}_0$ on the base $\mathcal{Q}_1(p,n)$ and is the measure $\tilde{\rho}_\omega$ on the fiber $\tilde{p}_0^{-1}(\omega)$. In other words, $d\tilde{\mu}_1(\eta) = d\tilde{\rho}_\omega(\eta)\, d\tilde{\mu}_0(\omega)$. Similarly, the space $\mathcal{MY}_1(p,n)$ carries a natural measure μ_1 such that $d\mu_1(\eta) = d\rho_\omega(\eta)\, d\mu_0(\omega)$. Notice that $\mu_1(\mathcal{MY}_1(p,n)) = \mu_0(\mathcal{MQ}_1(p,n)) < \infty$. The measure μ_1 is invariant under the action of the group $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{MY}_1(p,n)$.

Theorem 6.8 ([EM]) *The* $\mathrm{SL}(2,\mathbb{R})$ *action on* $\mathcal{MY}_1(p,n)$ *is ergodic with respect to the measure* μ_1 *on each connected component of* $\mathcal{MY}_1(p,n)$.

7 Periodic geodesics

Let M be a translation surface. Any geodesic joining a nonsingular point of M to itself is *periodic* (or *closed*) since it cannot change its direction. We only consider *primitive* periodic geodesics, that is, a periodic geodesic is a simple closed curve. Each unoriented periodic geodesic corresponds to two oriented periodic geodesics of the same length and of opposite directions. If a geodesic starting at a point $x \in M$ is periodic, then all geodesics starting at nearby points in the same direction are also periodic. Actually, each periodic geodesic belongs to a family of freely homotopic periodic geodesics of the same length and direction. If M is a torus without singular points, then this family fills the whole surface M. Otherwise the family fills a domain homeomorphic to an annulus. This domain is called a *cylinder* of periodic geodesics (or simply a *periodic cylinder*) since it is isometric to a cylinder $\mathbb{R}/l\mathbb{Z} \times (0,w)$, where $l, w > 0$. The number l is called the *length* or the *waist* of the periodic cylinder; it is equal to the length of periodic geodesics in the cylinder. w is called the *width* or the *height* of the cylinder. The cylinder is bounded by saddle connections parallel to its geodesics. The boundary of the cylinder is a union of two components. If geodesics in the cylinder are oriented then we can refer to them as the *left* and the *right* components. If M is a translation torus with one singular point then both components coincide and consist of a single saddle connection. Otherwise the left and the right components are different although they may share some saddle connections. A periodic cylinder is called *regular* if each component of its boundary consists of a single saddle connection or, equivalently, if bounding saddle connections are of the same length as periodic geodesics in the cylinder.

Each cylinder of unoriented periodic geodesics corresponds to two *oriented periodic cylinders*, i.e., cylinders of oriented periodic geodesics. By definition, the *direction* and the *holonomy vector* of an oriented periodic cylinder are the direction and the holonomy vector of an arbitrary periodic geodesic in the cylinder.

This section is devoted to the proofs of statements formulated in Section 1. We begin with Propositions 1.3 and 1.4.

Lemma 7.1 *(a) Let $h_1, h_2 \in H_1(M_p, Z_n; \mathbb{Z})$ be relative homology classes that are not multiples of the same homology class. Then for almost all $q \in H^1(M_p, Z_n; \mathbb{R}^2)$ vectors $q(h_1), q(h_2) \in \mathbb{R}^2$ have different lengths and directions.*

(b) Let $h_1', h_2' \in H_1(M_p, Z_n; \mathbb{Z})$ be another pair of relative homology classes. Let Λ be the euclidean area form on $\mathbb{R}^2$. Then $\Lambda(q(h_1'), q(h_2')) \neq \Lambda(q(h_1), q(h_2))$ for almost all $q \in H^1(M_p, Z_n; \mathbb{R}^2)$ unless $h_1' = b_{11}h_1 + b_{12}h_2$ and $h_2' = b_{21}h_1 + b_{22}h_2$, where $b_{ij} \in \mathbb{Q}$, $b_{11}b_{22} - b_{12}b_{21} = 1$.

Proof. Since h_1 and h_2 are not multiples of the same homology class, there exist $h_3, \dots, h_N \in H_1(M_p, Z_n; \mathbb{Z})$, $N = 2p+n-1$, such that $h_1, \dots, h_N$ is a basis for a finite index subgroup of $H_1(M_p, Z_n; \mathbb{Z})$. Then $H^1(M_p, Z_n; \mathbb{R}^2) \ni q \mapsto (q(h_1), \dots, q(h_N)) \in (\mathbb{R}^2)^N$ is an isomorphism of vector spaces. The set of pairs $(v_1, v_2) \in (\mathbb{R}^2)^2$ such that $v_1 = 0$ or $v_2 = 0$ or vectors v_1 and v_2 have the same length or direction is of zero Lebesgue measure. By Fubini's theorem, for almost all $q \in H^1(M_p, Z_n; \mathbb{R}^2)$ vectors $q(h_1)$ and $q(h_2)$ are nonzero vectors of different length and direction.

Any $h_1', h_2' \in H_1(M_p, Z_n; \mathbb{Z})$ are uniquely represented as $h_1' = b_{11}h_1 + \dots + b_{1N}h_N$ and $h_2' = b_{21}h_1 + \dots + b_{2N}h_N$, where $b_{ij} \in \mathbb{Q}$. Since $\Lambda(q(h_1), q(h_2))$ and $\Lambda(q(h_1'), q(h_2'))$ are quadratic forms of a vector $q \in H^1(M_p, Z_n; \mathbb{R}^2)$, we have either $\Lambda(q(h_1'), q(h_2')) \neq \Lambda(q(h_1), q(h_2))$ for almost all q or $\Lambda(q(h_1'), q(h_2')) = \Lambda(q(h_1), q(h_2))$ for all q. In the latter case it follows that $b_{ij} = 0$ for $j \geq 3$, that is, $h_1' = b_{11}h_1 + b_{12}h_2$ and $h_2' = b_{21}h_1 + b_{22}h_2$. Then $\Lambda(q(h_1'), q(h_2')) = (b_{11}b_{22} - b_{12}b_{21})\Lambda(q(h_1), q(h_2))$ for any q. For almost all q vectors $q(h_1)$ and $q(h_2)$ are nonzero and of different directions, hence $\Lambda(q(h_1), q(h_2)) \neq 0$. This implies $b_{11}b_{22} - b_{12}b_{21} = 1$. ∎

Proof of Propositions 1.3 and 1.4(a). Suppose $U \subset M_p$ is a domain bounded by disjoint saddle connections of a translation structure $\omega \in \Omega(p, n)$. Let $\gamma_1, \dots, \gamma_k$ be boundary saddle connections that are not slits, i.e., not surrounded by U. The orientation of M_p induces orientations of $\gamma_1, \dots, \gamma_k$. Assuming these orientations, the sum of the relative homology classes of $\gamma_1, \dots, \gamma_k$ in $H_1(M_p, Z_n; \mathbb{Z})$ is equal to zero. In particular, $\mathrm{hol}_\omega(\gamma_1) + \dots + \mathrm{hol}_\omega(\gamma_k) = 0$. The above argument easily implies that saddle connections belonging to the same component of boundary of a periodic cylinder do not divide the surface.

Let $[\gamma] \in H_1(M_p, Z_n; \mathbb{Z})$ be the relative homology class of a saddle connection γ of some $\omega \in \Omega(p, n)$. By Lemma 6.2, $[\gamma]$ is an element of a basis for $H_1(M_p, Z_n; \mathbb{Z})$. In particular, if $[\gamma] = bh$ for some $h \in H_1(M_p, Z_n; \mathbb{Z})$ and $b \in \mathbb{Z}$ then $b = \pm 1$. It follows that homology classes of distinct saddle connections γ, γ' of ω are multiples of the same homology class only if γ and γ' are *homologous*, i.e., they are disjoint and divide the surface M_p. Then Lemma 7.1(a) implies that for a generic translation surface any nonhomologous saddle connections are of different length and direction. By the above the boundary of any irregular periodic cylinder contains nonhomologous saddle connections, which are parallel. Therefore generic translation surfaces admit only regular periodic cylinders. Two periodic geodesics in different regular cylinders are homologous if and only if saddle connections bounding these cylinders are homologous. Since boundary saddle connections have the same length and direction as periodic geodesics in the cylinder, it follows that for a generic translation surface any nonhomologous periodic geodesics are of different length and direction. ∎

Proof of Proposition 1.4(b). Suppose C is a regular periodic cylinder of a translation structure $\omega \in \Omega(p,n)$, where $(p,n) \neq (1,1)$. Let γ be a saddle connection bounding C. Let γ' be a saddle connection that crosses C from one side to another and does not leave the cylinder. By Λ denote the euclidean area form on $\mathbb{R}^2$. Then the area of C is equal to $|\Lambda(\mathrm{hol}_\omega(\gamma), \mathrm{hol}_\omega(\gamma'))|$. Clearly, γ and γ' are disjoint and do not divide the surface. By Lemma 6.2, there exist saddle connections $\gamma_3, \dots, \gamma_N$ ($N = 2p + n - 1$) such that $\gamma, \gamma', \gamma_3, \dots, \gamma_N$ are also disjoint and do not divide the surface. The relative homology classes $[\gamma], [\gamma'], [\gamma_3], \dots, [\gamma_N]$ comprise a basis for $H_1(M_p, Z_n; \mathbb{Z})$. Let C_1 be a periodic cylinder of ω different from C. Let L be a periodic geodesic in C_1 and L' be a saddle connection crossing C_1. Then the area of C_1 is equal to $|\Lambda(\mathrm{hol}_\omega(L), \mathrm{hol}_\omega(L'))|$. Since $(p,n) \neq (1,1)$ and C is a regular cylinder, the geodesic L is not contained in the closure of C. If L is disjoint from C then its homology class $[L]$ is a linear combination of $[\gamma], [\gamma_3], \dots, [\gamma_N]$. Otherwise L is freely homotopic to the sum of saddle connections $L_1, L'_1, \dots, L_k, L'_k$ such that $L_1, \dots, L_k$ are disjoint from C while $L'_1, \dots, L'_k$ are contained in the closure of C. The relative homology classes of $L_1, \dots, L_k$ are linear combinations of $[\gamma], [\gamma_3], \dots, [\gamma_N]$ while the relative homology classes of $L'_1, \dots, L'_k$ are linear combinations of $[\gamma]$ and $[\gamma']$. Note that projections of holonomy vectors $\mathrm{hol}_\omega(L_i)$, $\mathrm{hol}_\omega(L'_i)$, $i = 1, \dots, k$, on the direction orthogonal to γ are all nonzero and of the same sign. It follows that the sum of homology classes of $L_1, \dots, L_k$ is not a multiple of $[\gamma]$. Thus $[L]$ can be a linear combination of $[\gamma]$ and $[\gamma']$ only if it is a multiple of $[\gamma]$. But then the cylinders C and C_1 are parallel and disjoint, hence the homology class of L' is not a linear combination of $[\gamma]$ and $[\gamma']$.

By the above and Lemma 7.1(b), for a translation surface in a generic isomorphy class $\omega \in \mathcal{MQ}(p,n)$, $(p,n) \neq (1,1)$, all regular periodic cylinders are of different area. It remains to recall that generic translation surfaces do not contain irregular cylinders. ∎

We proceed to the proofs of Theorems 1.5–1.10. Let M be a translation surface of genus p with n singular points ($p, n \geq 1$). We assign to M a sequence $V_1(M)$ of vectors in $\mathbb{R}^2$. These are the holonomy vectors of oriented periodic cylinders on M. Note that any (unoriented) periodic cylinder corresponds to two oriented cylinders with opposite holonomy vectors. Both vectors are supposed to be in the sequence. If a vector is the holonomy vector of $k > 1$ distinct periodic cylinders, it is to appear k times in $V_1(M)$. By Theorem 1.1(c), the sequence $V_1(M)$ tends to infinity. To make $V_1(M)$ an element of the set $\mathcal{V}$ defined in Section 3, we have to equip vectors with weights. Let all weights be equal to 1. Another choice is to let the weight of the holonomy vector of a cylinder be equal to the area of the cylinder. Then we obtain a different element of $\mathcal{V}$ that is denoted by $V_2(M)$. Further, for any $x \in M$ we define $V_3(M, x) \in \mathcal{V}$ to be the sequence of holonomy vectors of oriented periodic geodesics on M passing through the point x. By definition, all weights of $V_3(M,x)$ are equal to 1. Now the growth functions N_1, N_2, N_3, and N_4 defined in Section 1 can be expressed as follows: $N_1(M,R) = N_{V_1(M)}(R)/2$, $N_2(M,R) = N_{V_2(M)}(R)/2$, $N_3(M,x,R) = N_{V_3(M,x)}(R)/2$, and $N_4(M,\sigma,R) = N_{W_{(\sigma,\infty)}V_2(M)}(R)/2$ for all $R > 0$.

Let $\mathcal{C}$ be a connected component of $\mathcal{MQ}_1(p,n)$. Let $p_0 : \mathcal{MY}(p,n) \to \mathcal{MQ}(p,n)$ be the natural projection. Then $Y = p_0^{-1}(\mathcal{C})$ is a connected component of $\mathcal{MY}_1(p,n)$. The sequences $V_1(M)$ and $V_2(M)$ do not change when M is replaced by an isomorphic translation surface. The sequence $V_3(M,x)$ does not change when (M,x) is replaced by a pair representing the same equivalence class in $\mathcal{MY}(p,n)$. Hence the assignments $M \mapsto V_1(M)$, $M \mapsto V_2(M)$, and $(M,x) \mapsto V_3(M,x)$ give rise to well-defined maps $V_1 : \mathcal{C} \to \mathcal{V}$, $V_2 : \mathcal{C} \to \mathcal{V}$, and $V_3 : Y \to \mathcal{V}$. Recall that there are continuous actions of the group $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{C}$ and Y. By Theorems 6.6 and 6.7, the Borel measure μ_0 induced by the canonical volume element on $\mathcal{C}$ is finite and the $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{C}$ is ergodic relative to this measure. The measure μ_1 on Y is also finite and the $\mathrm{SL}(2,\mathbb{R})$ action on Y is ergodic by Theorem 6.8. Thus the results of Sections 3 and 4 apply to the maps V_1, V_2, and V_3. Let us check whether the conditions formulated in Sections 3 and 4 hold for these maps.

Proposition 7.2 *The maps V_1, V_2, and V_3 satisfy conditions (0), (A), (B$'$), (C), and (E). The map V_2 satisfies condition (0$'$).*

Proof. By definition of the $\mathrm{SL}(2,\mathbb{R})$ actions on $\mathcal{C}$, Y, and $\mathcal{V}$, the maps V_1, V_2, and V_3 satisfy condition (A). Condition (E) holds trivially.

Let $\mathcal{S}(p,n)$ be the set of free homotopy classes of simple closed oriented curves in $M_p \setminus Z_n$. Given $\gamma \in \mathcal{S}(p,n)$, let $U(\gamma) \subset \Omega(p,n)$ be the set of translation structures that admit a periodic geodesic in the homotopy class γ. By $U_1(\gamma)$ denote the set of pairs $(\omega,x) \in \Omega(p,n) \times M_p$ such that some periodic geodesic of the translation structure ω passing through the point x is in the homotopy class γ. Given $\omega \in U(\gamma)$, all periodic geodesics of ω that belong to the homotopy class γ comprise one periodic cylinder. Let $a_\gamma(\omega)$ denote the area of this cylinder. For any $\omega \notin U(\gamma)$ let $a_\gamma(\omega) = 0$. The sets $U(\gamma)$ and $U_1(\gamma)$ are invariant under the $H_0(p,n)$ actions on $\Omega(p,n)$ and $\Omega(p,n) \times M_p$, respectively. Therefore we consider $U(\gamma)$ as a subset of $\mathcal{Q}(p,n)$ and $U_1(\gamma)$ as a subset of $\mathcal{Y}(p,n)$. It is easy to see that $U(\gamma)$ and $U_1(\gamma)$ are open sets. The map $\mathcal{Q}(p,n) \ni \omega \mapsto \mathrm{hol}_\omega(\gamma) \in \mathbb{R}^2$ is well-defined and continuous. a_γ descends to a function on $\mathcal{Q}(p,n)$, which is also continuous. Indeed, let $\omega_0 \in U(\gamma)$ and ω_0' be a translation structure in the isotopy class ω_0. Take a saddle connection γ_1 of ω_0' that crosses the cylinder of periodic geodesics with homotopy γ and does not leave this cylinder. Let $[\gamma_1] \in H_1(M_p, Z_n; \mathbb{Z})$ be the homology class of γ_1. Then $a_\gamma(\omega_0) = |\Lambda(\mathrm{hol}_{\omega_0}(\gamma), \mathrm{hol}_{\omega_0}([\gamma_1]))|$, where Λ is the euclidean area form on $\mathbb{R}^2$. If the cylinder is regular then $a_\gamma(\omega) = |\Lambda(\mathrm{hol}_\omega(\gamma), \mathrm{hol}_\omega([\gamma_1]))|$ for all $\omega \in \mathcal{Q}(p,n)$ in a neighborhood of ω_0. In the general case $a_\gamma(\omega) \leq |\Lambda(\mathrm{hol}_\omega(\gamma), \mathrm{hol}_\omega([\gamma_1]))|$ but we can choose several saddle connections $\gamma_1, \dots, \gamma_k$ of ω_0' so that

$$a_\gamma(\omega) = \min_{1 \leq j \leq k} |\Lambda(\mathrm{hol}_\omega(\gamma), \mathrm{hol}_\omega([\gamma_j]))|$$

for all ω close to ω_0.

Let $\pi_0 : \mathcal{Q}(p,n) \to \mathcal{MQ}(p,n)$, $\pi_1 : \mathcal{Y}(p,n) \to \mathcal{MY}(p,n)$, and $\tilde{p}_0 : \mathcal{Y}(p,n) \to \mathcal{Q}(p,n)$ be the canonical projections. Then for any $\omega \in \pi_0^{-1}(\mathcal{C})$, $\eta \in \pi_1^{-1}(Y)$, $\psi \in C_c(\mathbb{R}^2)$, and $f \in C_c(\mathbb{R}^+)$ we have

$$\begin{aligned}
\Phi[V_1(\pi_0(\omega))](\psi) &= \sum_{\gamma\in\mathcal{S}(p,n)} \chi_{U(\gamma)}(\omega)\,\psi(\mathrm{hol}_\omega(\gamma)),\\
\Psi[V_2(\pi_0(\omega))](f,\psi) &= \sum_{\gamma\in\mathcal{S}(p,n)} f(a_\gamma(\omega))\,\psi(\mathrm{hol}_\omega(\gamma)),\\
\Phi[V_3(\pi_1(\eta))](\psi) &= \sum_{\gamma\in\mathcal{S}(p,n)} \chi_{U_1(\gamma)}(\eta)\,\psi(\mathrm{hol}_{\tilde{p}_0(\eta)}(\gamma)).
\end{aligned}$$

All three sums are locally finite. It follows that the functions $\pi_0^{-1}(\mathcal{C}) \ni \omega \mapsto \Phi[V_1(\pi_0(\omega))](\psi)$ and $\pi_1^{-1}(Y) \ni \eta \mapsto \Phi[V_3(\pi_1(\eta))](\psi)$ are Borel, while the function $\pi_0^{-1}(\mathcal{C}) \ni \omega \mapsto \Psi[V_2(\pi_0(\omega))](f,\psi)$ is continuous. Then the functions $\mathcal{C} \ni \omega \mapsto \Phi[V_1(\omega)](\psi)$ and $Y \ni \eta \mapsto \Phi[V_3(\eta)](\psi)$ are also Borel and the function $\mathcal{C} \ni \omega \mapsto \Psi[V_2(\omega)](f,\psi)$ is also continuous. Thus the maps V_1 and V_3 satisfy condition (0) while the map V_2 satisfies condition $(0'')$. By Lemma 4.2, V_2 satisfies conditions (0) and $(0')$ as well.

For any $\omega \in \mathcal{C}$, let $s(\omega)$ denote the length of the shortest saddle connection of translation structures in the class ω. The function $\omega \mapsto s(\omega)$ is continuous and bounded on $\mathcal{C}$. Therefore the upper estimate in Theorem 1.2(c) implies the map V_1 satisfies condition (B$'$). To verify condition (C), we need the following theorem.

Theorem 7.3 ([EM]) *(a) Given $\epsilon > 0$, there exist $C_\epsilon > 0$ and $\kappa > 0$ such that*

$$N_{V_1(\omega)}(R) \le C_\epsilon \left(\frac{R}{s(\omega)}\right)^{1+\epsilon}$$

for any $R < \kappa$ and any $\omega \in \mathcal{C}$.

(b) For any $\beta \in [1,2)$ the function $s^{-\beta}$ belongs to the space $L^1(\mathcal{C},\mu_0)$.

Theorem 7.3 implies that condition (C) holds for the map V_1. Let $\omega \in \mathcal{C}$. By definition, $N_{V_2(\omega)}(R) \le N_{V_1(\omega)}(R)$ for any $R > 0$, and $N_{V_3(\eta)}(R) \le N_{V_1(\omega)}(R)$ for any $\eta \in p_0^{-1}(\omega)$ and any $R > 0$. It follows that conditions (B$'$) and (C) are satisfied by the maps V_2 and V_3 whenever these conditions are satisfied by V_1. ∎

Proof of Theorems 1.5, 1.6(a), and 1.7. By Proposition 7.2, the maps V_1, V_2, and V_3 satisfy conditions (0), (A), (B), and (C). Let $c_1(\mathcal{C})$, $c_2(\mathcal{C})$, and $c_3(\mathcal{C})$ be nonnegative numbers such that $2\pi^{-1}c_1(\mathcal{C})$, $2\pi^{-1}c_2(\mathcal{C})$, and $2\pi^{-1}c_3(\mathcal{C})$ are the Siegel-Veech constants of the pairs (V_1,μ_0), (V_2,μ_0), and (V_3,μ_1), respectively. By Part II of Theorem 3.2, for μ_0-almost every $\omega \in \mathcal{C}$ we have

$$\lim_{R\to\infty} N_{V_1(\omega)}(R)/R^2 = 2c_1(\mathcal{C}), \tag{8}$$

$$\lim_{R\to\infty} N_{V_2(\omega)}(R)/R^2 = 2c_2(\mathcal{C}), \tag{9}$$

and for μ_1-almost every $\eta \in Y$,

$$\lim_{R\to\infty} N_{V_3(\eta)}(R)/R^2 = 2c_3(\mathcal{C}). \tag{10}$$

The positivity of $c_1(\mathcal{C})$ and $c_2(\mathcal{C})$ follows from Proposition 3.3 and Theorem 1.1(a). Let us show that $c_3(\mathcal{C}) = c_2(\mathcal{C})$. By Part I of Theorem 3.2, we have

$$\begin{aligned} \frac{1}{\mu_0(\mathcal{C})}\int_{\mathcal{C}} N_{V_2(\omega)}(1)\, d\mu_0(\omega) &= 2c_2(\mathcal{C}), \\ \frac{1}{\mu_1(Y)}\int_{Y} N_{V_3(\eta)}(1)\, d\mu_1(\eta) &= 2c_3(\mathcal{C}). \end{aligned}$$

For any $\omega \in \mathcal{C}$, let ρ_ω denote the Borel measure on the fiber $p_0^{-1}(\omega)$ induced by translation structures in the equivalence class ω. It is easy to observe that

$$N_{V_2(\omega)}(R) = \int_{p_0^{-1}(\omega)} N_{V_3(\eta)}(R)\, d\rho_\omega(\eta)$$

for all $R > 0$. Then

$$\int_Y N_{V_3(\eta)}(1)\, d\mu_1(\eta) = \int_{\mathcal{C}}\int_{p_0^{-1}(\omega)} N_{V_3(\eta)}(1)\, d\rho_\omega(\eta)\, d\mu_0(\omega) = \int_{\mathcal{C}} N_{V_2(\omega)}(1)\, d\mu_0(\omega),$$

besides,

$$\mu_1(Y) = \int_{\mathcal{C}} \rho_\omega(p_0^{-1}(\omega))\, d\mu_0(\omega) = \mu_0(\mathcal{C}).$$

Hence, $c_3(\mathcal{C}) = c_2(\mathcal{C})$.

By the above there exists a Borel set $U \subset \mathcal{C}$, $\mu_0(U) = \mu_0(\mathcal{C})$, such that for any $\omega \in U$ the relations (8) and (9) hold, and, moreover, the relation (10) holds for ρ_ω-almost all $\eta \in p_0^{-1}(\omega)$. Let M be a translation surface in an isomorphy class $\omega \in U$. Then $N_1(M,R)/R^2 \to c_1(\mathcal{C})$ and $N_2(M,R)/R^2 \to c_2(\mathcal{C})$ as $R \to \infty$, and for almost all $x \in M$, $N_3(M,x,R)/R^2 \to c_2(\mathcal{C})$ as $R \to \infty$. ∎

To prove Theorems 1.6(b), 1.8, and 1.9, we need the following proposition, which will be proved in Section 8.

Proposition 7.4 *For any $\sigma \in [0,1)$ and $\epsilon > 0$ let*

$$b(\sigma,\epsilon) = \frac{1}{\mu_0(\mathcal{C})}\int_{\mathcal{C}} N_{W_{(\sigma,\infty)}V_2(\omega)}(\epsilon)\, d\mu_0(\omega). \tag{11}$$

Then for any $\sigma \in [0,1)$,

$$\lim_{\epsilon\to 0}\frac{b(\sigma,\epsilon)}{b(0,\epsilon)} = (1-\sigma)^{m_{\mathcal{C}}-1},$$

where $m_{\mathcal{C}} = 2p - 2 + n$.

Proof of Theorems 1.6(b) and 1.8. By Proposition 7.2, the map V_2 satisfies conditions (0′), (A), and (B), hence Proposition 4.3 applies to it. Let λ be the Siegel-Veech measure of the pair (V_2, μ_0). Since the map $V_1 = W_{(0,\infty)}V_2$ satisfies condition (B), λ is a finite measure on $\mathbb{R}^+$. For any $a \geq 0$ the Siegel-Veech constant of $(W_{(a,\infty)}V_2, \mu_0)$ is equal to $\lambda((a, \infty))$. By Part I of Theorem 3.2, $b(a, \epsilon) = \pi\epsilon^2\lambda((a, \infty))$ for all $\epsilon > 0$ and $a \in [0, 1)$. Then it follows from Proposition 7.4 that $\lambda((a, \infty)) = (1 - a)^{m_{\mathcal{C}}-1}\lambda(\mathbb{R}^+)$ for all $a \in [0, 1)$. For any $\omega \in \mathcal{C}$ the sequence $V_2(\omega)$ contains no vectors with weights greater than 1, therefore $\lambda((1, \infty)) = 0$. Let c_0 be the Siegel-Veech constant of (V_2, μ_0). If $m_{\mathcal{C}} = 1$ then $\lambda(\{1\}) = \lambda(\mathbb{R}^+)$, hence $c_0 = \lambda(\mathbb{R}^+)$. If $m_{\mathcal{C}} > 1$ then

$$c_0 = \int_0^\infty t\, d\lambda(t) = -\lambda(\mathbb{R}^+) \int_0^1 t\, d(1 - t)^{m_{\mathcal{C}}-1} = \lambda(\mathbb{R}^+)/m_{\mathcal{C}}.$$

By Part II of Theorem 3.2, for any $a \in [0, 1)$ we have

$$\lim_{R\to\infty} N_{W_{(a,\infty)}V_2(\omega)}(R)/R^2 = \pi(1 - a)^{m_{\mathcal{C}}-1}\lambda(\mathbb{R}^+) \tag{12}$$

for μ_0-almost all $\omega \in \mathcal{C}$. Also, for μ_0-almost all $\omega \in \mathcal{C}$,

$$\lim_{R\to\infty} N_{V_2(\omega)}(R)/R^2 = \pi m_{\mathcal{C}}^{-1}\lambda(\mathbb{R}^+). \tag{13}$$

Let $U \subset \mathcal{C}$ be a Borel set such that $\mu_0(U) = \mu_0(\mathcal{C})$, the relation (13) holds for any $\omega \in U$, and the relation (12) holds for any $\omega \in U$ and any rational $a \in [0, 1)$. Since $N_{W_{(a,\infty)}V_2(\omega)}(R)$ is a nonincreasing function of a, it follows that the relation (12) holds for any $\omega \in U$ and any $a \in [0, 1)$. Let M be a translation surface in an isomorphy class $\omega \in U$. Then $N_2(M, R)/R^2 \to \pi m_{\mathcal{C}}^{-1}\lambda(\mathbb{R}^+)/2$ as $R \to \infty$, and for any $a \in [0, 1)$, $N_4(M, a, R)/R^2 \to \pi(1 - a)^{m_{\mathcal{C}}-1}\lambda(\mathbb{R}^+)/2$ as $R \to \infty$. By Theorems 1.5 and 1.6(a), $\pi\lambda(\mathbb{R}^+)/2 = c_1(\mathcal{C})$ and $\pi m_{\mathcal{C}}^{-1}\lambda(\mathbb{R}^+)/2 = c_2(\mathcal{C})$. It follows that $c_2(\mathcal{C}) = c_1(\mathcal{C})/m_{\mathcal{C}}$. ∎

Proof of Theorems 1.9 and 1.10. By Proposition 7.2, the map V_2 satisfies conditions (0′), (A), (B), (C), and (E). Besides, the Siegel-Veech constant of (V_2, μ_0) is nonzero and the map $W_{(0,\infty)}V_2 = V_1$ satisfies condition (B). Therefore Propositions 4.4 and 4.6 apply to V_2. Let λ be the Siegel-Veech measure of the pair (V_2, μ_0). As shown in the previous proof, $\lambda((a, \infty)) = (1 - a)^{m_{\mathcal{C}}-1}\lambda(\mathbb{R}^+)$ for any $a \in [0, 1)$ and $\lambda((1, \infty)) = 0$. In particular, we can consider λ as a Borel measure on $[0, 1]$. Then the normalized measure $(\lambda(\mathbb{R}^+))^{-1}\lambda$ coincides with the measure $\lambda_{m_{\mathcal{C}}}$ defined in Section 1. Since for any $\omega \in \mathcal{C}$ the sequence $V_2(\omega)$ contains no vectors with weights greater than 1, we can consider the measures $\tilde{\alpha}_{V_2(\omega),R}$ and $\widetilde{D}_{V_2(\omega),R}$ defined in Section 4 as measures on $[0, 1]$ and $S^1 \times [0, 1]$, respectively. Then it follows from Propositions 4.4 and 4.6 that for μ_0-almost all $\omega \in \mathcal{C}$ we have the following weak convergence of measures:

$$\lim_{R\to\infty} \tilde{\alpha}_{V_2(\omega),R} = \lambda_{m_{\mathcal{C}}},$$

$$\lim_{R\to\infty} \widetilde{D}_{V_2(\omega),R} = \mathfrak{m}_1 \times \lambda_{m_{\mathcal{C}}}.$$

The maps V_1 and V_3 satisfy conditions (0), (A), (B), (C), and (E). The Siegel-Veech constants of the pairs (V_1, μ_0) and (V_3, μ_1) are nonzero. Then Proposition 4.5 implies that for μ_0-almost all $\omega \in \mathcal{C}$ and for μ_1-almost all $\eta \in Y$ the measures $\tilde{\delta}_{V_1(\omega),R}$ and $\tilde{\delta}_{V_3(\eta),R}$ on S^1 weakly converge to normalized Lebesgue measure $\mathfrak{m}_1$ as $R \to \infty$.

By the above there exists a Borel set $U \subset \mathcal{C}$, $\mu_0(U) = \mu_0(\mathcal{C})$, such that for any $\omega \in U$ the measures $\tilde{\alpha}_{V_2(\omega),R}$, $\tilde{\delta}_{V_1(\omega),R}$, and $\widetilde{D}_{V_2(\omega),R}$ weakly converge to $\lambda_{m_\mathcal{C}}$, $\mathfrak{m}_1$, and $\mathfrak{m}_1 \times \lambda_{m_\mathcal{C}}$, respectively, as $R \to \infty$, and for ρ_ω-almost all $\eta \in p_0^{-1}(\omega)$ the measures $\tilde{\delta}_{V_3(\eta),R}$ weakly converge to $\mathfrak{m}_1$ as $R \to \infty$. Let M be a translation surface in an isomorphy class $\omega \in U$. Then the measures $\tilde{\alpha}_{M,R}$, $\tilde{\delta}_{M,R}$, and $\widetilde{D}_{M,R}$ defined in Section 1 coincide with $\tilde{\alpha}_{V_2(\omega),R}$, $\tilde{\delta}_{V_1(\omega),R}$, and $\widetilde{D}_{V_2(\omega),R}$, respectively. Further, for any $x \in M$ the measure $\tilde{\delta}_{M,x,R}$ coincides with $\tilde{\delta}_{V_3(\eta),R}$, where $\eta \in p_0^{-1}(\omega)$ is the equivalence class of (M, x). Theorems 1.9 and 1.10 follow. ∎

8 Areas of periodic cylinders

For any positive integer n define sets

$$S_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_i > 0,\ t_1 + \dots + t_n \le 1\},$$

$$S_n^* = \{(t, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid t > 0,\ t_i > 0,\ t^2 + t_1 + \dots + t_n \le 1\}.$$

Further, for any $\sigma \in [0,1)$ and $i \in \{1, \dots, n\}$ define their subsets

$$S_n(\sigma, i) = \{(t_1, \dots, t_n) \in S_n \mid t_i > \sigma(t_1 + \dots + t_n)\},$$

$$S_n^*(\sigma, i) = \{(t, t_1, \dots, t_n) \in S_n^* \mid t_i > \sigma(t^2 + t_1 + \dots + t_n)\}.$$

Note that $S_n(0, i) = S_n$, $S_n^*(0, i) = S_n^*$. Now let

$$J_{n,0}(\sigma, i) = \int_{S_n(\sigma,i)} dt_1 \dots dt_n$$

and for any positive integer k,

$$J_{n,k}(\sigma, i) = \int_{S_n^*(\sigma,i)} dt^{2k}\, dt_1 \dots dt_n.$$

Lemma 8.1 *For any integers $n > 0$ and $k \ge 0$, any $i \in \{1, \dots, n\}$, and any $\sigma \in [0, 1)$,*

$$\frac{J_{n,k}(\sigma, i)}{J_{n,k}(0, i)} = (1 - \sigma)^{n+k-1}.$$

Proof. Since $J_{n,k}(\sigma, i) = J_{n,k}(\sigma, 1)$ for any $i \in \{1, \dots, n\}$, we can assume without loss of generality that $i = 1$. The case $n = 1$, $k = 0$ is trivial as $S_1(\sigma, 1) = S_1$ for any $\sigma \in [0, 1)$. Given $\sigma \in (0, 1)$ and $b > 0$, let

$$I_b(\sigma) = \int_0^1 \Big(\min(1 - t, t/\sigma - t)\Big)^b\, dt.$$

Then

$$I_b(\sigma) = \int_0^\sigma (t/\sigma - t)^b\, dt + \int_\sigma^1 (1-t)^b\, dt =$$

$$\frac{1}{b+1}\Big(\sigma^{b+1}(\sigma^{-1}-1)^b + (1-\sigma)^{b+1}\Big) = \frac{(1-\sigma)^b}{b+1}.$$

Suppose $n > 1$ and $\sigma \in (0,1)$. Then

$$J_{n,0}(\sigma,1) = \int_0^1 dt_1 \int_{\min(1-t_1, t_1/\sigma - t_1)S_{n-1}} dt_2 \dots dt_n = I_{n-1}(\sigma) \int_{S_{n-1}} dt_2 \dots dt_n.$$

Now suppose $n > 0$ and $\sigma \in (0,1)$. Then $(t, t_1, \dots, t_n) \in S_n^*(\sigma, 1)$ if and only if $(t_1, \dots, t_n) \in S_n(\sigma, 1)$, $t > 0$, $t \le (1 - t_1 - \dots - t_n)^{1/2}$, and $t < (t_1/\sigma - t_1 - \dots - t_n)^{1/2}$. Hence for any integer $k > 0$,

$$J_{n,k}(\sigma,1) = \int_{S_n(\sigma,1)} \Big(\min(1, t_1/\sigma) - t_1 - \dots - t_n\Big)^k dt_1 \dots dt_n.$$

In particular, $J_{1,k}(\sigma,1) = I_k(\sigma)$. If $n > 1$ then

$$J_{n,k}(\sigma,1) = \int_0^1 dt_1 \int_{\min(1-t_1, t_1/\sigma - t_1)S_{n-1}} \Big(\min(1, t_1/\sigma) - t_1 - \dots - t_n\Big)^k dt_2 \dots dt_n =$$

$$I_{n+k-1}(\sigma) \int_{S_{n-1}} (1 - t_2 - \dots - t_n)^k\, dt_2 \dots dt_n.$$

By the above for any integers $n > 0$ and $k \ge 0$ there exists $C_{n,k} > 0$ such that $J_{n,k}(\sigma,1) = (1-\sigma)^{n+k-1} C_{n,k}$ for all $\sigma \in (0,1)$. Since $J_{n,k}(\sigma,1) \to J_{n,k}(0,1)$ as $\sigma \to 0$, it follows that $C_{n,k} = J_{n,k}(0,1)$. ∎

Let $\omega^{(n)} \in \Omega(1,n)$ be a translation structure with $n \ge 1$ singular points on the torus M_1 such that for some $v \in S^1$ there are n cylinders of periodic geodesics of $\omega^{(n)}$ going in direction v (in general, the number of cylinders may be less than n). Let $x_1, \dots, x_n$ be singular points of $\omega^{(n)}$ and $C_1, \dots, C_n$ be periodic cylinders with direction v. By L_i ($1 \le i \le n$) denote the saddle connection going out of x_i in direction v. We assume that the singular points and the cylinders are named so that L_i separates cylinders C_i and C_{i-1} (where by definition $C_0 = C_n$) and, furthermore, C_i lies to the right of L_i (with respect to the direction of L_i). Let γ_i be a saddle connection that crosses the cylinder C_i joining x_i to x_{i+1} (by definition, $x_{n+1} = x_1$). Note that the holonomy vector of γ_i and v induce the standard orientation in $\mathbb{R}^2$. Among all saddle connections that join x_i to x_{i+1} by crossing C_i, only one is disjoint from γ_i. Let γ_i' denote this saddle connection. It is easy to see that L_i, γ_i, γ_i', $i = 1, \dots, n$, is a maximal set of disjoint saddle connections of $\omega^{(n)}$. Let $\tau^{(n)} \in \mathcal{T}(1,n)$ be the affine equivalence class of the corresponding triangulation. For future references, we denote by $L^{(n)}$ the relative homotopy class of L_1, which is regarded as an edge of $\tau^{(n)}$. For any translation structure in an isotopy class $\omega \in N(\tau^{(n)})$ there are n cylinders of homologous periodic geodesics freely homotopic in $M_1 \setminus \{x_1, \dots, x_n\}$

to geodesics in the cylinders $C_1, \ldots, C_n$. Finally, by $P^{(n)}$ denote the set of $\omega \in N(\tau^{(n)}) \subset \mathcal{Q}(1,n)$ such that for any i, $1 \le i \le n$, we have $\mathrm{hol}_\omega(\gamma_i) = b_i \,\mathrm{hol}_\omega(L_i) + v_i$, where $0 < b_i < 1$, $v_i \in \mathbb{R}^2$ is orthogonal to $\mathrm{hol}_\omega(L_i)$, and the pair of vectors v_i and $\mathrm{hol}_\omega(L_i)$ induces the standard orientation in $\mathbb{R}^2$.

For any translation structure ω' on a surface M let $a(\omega')$ denote the area of M with respect to ω'. For any element ω of $\mathcal{Q}(p_0, n_0)$ or $\mathcal{MQ}(p_0, n_0)$ ($p_0, n_0 \ge 1$) let $a(\omega)$ denote the area of the surface M_{p_0} with respect to translation structures in the equivalence class ω. Further, for any translation structure ω' in an isotopy class $\omega \in N(\tau^{(n)})$ and any integer i, $1 \le i \le n$, let $a_i(\omega')$ denote the area of the periodic cylinder of ω' homotopic to C_i. The area $a_i(\omega')$ does not depend on the choice of $\omega' \in \omega$ and we let $a_i(\omega) = a_i(\omega')$. Obviously, $a_1(\omega) + \cdots + a_n(\omega) = a(\omega)$.

Given $U \subset \mathbb{R}^2$, let $P^{(n)}(U)$ be the set of $\omega \in P^{(n)}$ such that $\mathrm{hol}_\omega(L_1) \in U$ and $a(\omega) \le 1$. For any $i \in \{1, \ldots, n\}$ and any $\sigma \in [0,1)$ let $P^{(n)}_{\sigma,i}(U)$ be the set of $\omega \in P^{(n)}(U)$ such that $a_i(\omega) > \sigma a(\omega)$. Given sets $U_j \subset \mathcal{Q}_1(p_j, n_j)$, $1 \le j \le k$, let $P^{(n)}(U; U_1, \ldots, U_k)$ denote the set of $(\omega, \omega_1, \ldots, \omega_k) \in \mathcal{Q}(1,n) \times \mathcal{Q}(p_1, n_1) \times \ldots \times \mathcal{Q}(p_k, n_k)$ such that $\omega \in P^{(n)}(U)$, each ω_j is represented as $t_j \omega'_j$, where $\omega'_j \in U_j$ and $0 < t_j \le 1$, and $a(\omega) + a(\omega_1) + \cdots + a(\omega_k) \le 1$. For any $i \in \{1, \ldots, n\}$ and $\sigma \in [0,1)$ let $P^{(n)}_{\sigma,i}(U; U_1, \ldots, U_k)$ be the set of $(\omega, \omega_1, \ldots, \omega_k) \in P^{(n)}(U; U_1, \ldots, U_k)$ such that $a_i(\omega) > \sigma(a(\omega) + a(\omega_1) + \cdots + a(\omega_k))$.

Lemma 8.2 *(a) Suppose $U \subset \mathbb{R}^2$ is a nonempty open bounded set. Then for any $\sigma \in [0,1)$ and $i \in \{1, \ldots, n\}$,*

$$\frac{\mu_0(P^{(n)}_{\sigma,i}(U))}{\mu_0(P^{(n)}(U))} = (1-\sigma)^{n-1},$$

where μ_0 denotes the canonical measure on $\mathcal{Q}(1,n)$.

(b) Suppose $U \subset \mathbb{R}^2$ and $U_j \subset \mathcal{Q}_1(p_j, n_j)$, $1 \le j \le k$, are nonempty open sets of finite measure. Then for any $\sigma \in [0,1)$ and $i \in \{1, \ldots, n\}$,

$$\frac{\mu(P^{(n)}_{\sigma,i}(U; U_1, \ldots, U_k))}{\mu(P^{(n)}(U; U_1, \ldots, U_k))} = (1-\sigma)^{K-2},$$

where $K = n + 1 + \sum_{j=1}^{k}(2p_j + n_j - 1)$ is half of the dimension of $\mathcal{Q}(1,n) \times \mathcal{Q}(p_1, n_1) \times \ldots \times \mathcal{Q}(p_k, n_k)$ and μ denotes the product of the canonical measures on $\mathcal{Q}(1,n), \mathcal{Q}(p_1, n_1), \ldots, \mathcal{Q}(p_k, n_k)$.

Proof. The homology classes of saddle connections $L_1, \gamma_1, \ldots, \gamma_n$ form a basis for $H_1(M_1, \{x_1, \ldots, x_n\}; \mathbb{Z})$. The map $F_1 : N(\tau^{(n)}) \to (\mathbb{R}^2)^{n+1}$ defined by $F_1(\omega) = (\mathrm{hol}_\omega(L_1), \mathrm{hol}_\omega(\gamma_1), \ldots, \mathrm{hol}_\omega(\gamma_n))$ is a volume preserving homeomorphism of $N(\tau^{(n)})$ onto its image. Given a nonzero $v \in \mathbb{R}^2$, let g_v denote a unique element of $\mathrm{SL}(2,\mathbb{R})$ such that $g_v v = (0,1)$ and $g_v u = (1,0)$ for some u orthogonal to v. Define a transformation F_2 of the set $(\mathbb{R}^2 \setminus \{(0,0)\}) \times (\mathbb{R}^2)^n$ by $F_2(v, v_1, \ldots, v_n) = (v, g_v v_1, \ldots, g_v v_n)$. F_2 is a homeomorphism preserving Lebesgue measure. Suppose $(v, v_1, \ldots, v_n) =$

$F_2(F_1(\omega))$ for some $\omega \in P^{(n)}$. Then the first coordinate of the vector $v_i \in \mathbb{R}^2$, $1 \le i \le n$, is equal to $a_i(\omega)$ while the second coordinate lies between 0 and 1. Finally, define a map $F_3 : (\mathbb{R}^2)^{n+1} \to \mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^n$ by $F_3((v_{01}, v_{02}), (v_{11}, v_{12}), \dots, (v_{n1}, v_{n2})) = ((v_{01}, v_{02}), (v_{11}, \dots, v_{n1}), (v_{12}, \dots, v_{n2}))$. F_3 is a linear map preserving Lebesgue measure. It is easy to observe that $F_3 \circ F_2 \circ F_1(P^{(n)}(U)) = (U \setminus \{(0,0)\}) \times S_n \times (0,1)^n$ while $F_3 \circ F_2 \circ F_1(P^{(n)}_{\sigma,i}(U)) = (U \setminus \{(0,0)\}) \times S_n(\sigma, i) \times (0,1)^n$ for all σ and i. Therefore

$$\frac{\mu_0(P^{(n)}_{\sigma,i}(U))}{\mu_0(P^{(n)}(U))} = \frac{\mathfrak{m}(U) J_{n,0}(\sigma, i)}{\mathfrak{m}(U) J_{n,0}(0, i)} = (1-\sigma)^{n-1}$$

by Lemma 8.1.

For any $t \in (0,1]$ let U^*_t denote the set of $(\omega_1, \dots, \omega_k) \in \mathcal{Q}(p_1, n_1) \times \ldots \times \mathcal{Q}(p_k, n_k)$ such that each ω_j is represented as $t_j \omega'_j$, where $\omega'_j \in U_j$ and $0 < t_j \le 1$, and $a(\omega_1) + \dots + a(\omega_k) \le t^2$. Since $a(t\omega') = t^2 a(\omega')$ for any translation structure ω', it follows that $U^*_t = tU^*_1$, where by definition $t(\omega_1, \dots, \omega_k) = (t\omega_1, \dots, t\omega_k)$ for all $\omega_j \in \mathcal{Q}(p_j, n_j)$, $1 \le j \le k$. Denote by μ^* the product of the canonical measures on $\mathcal{Q}(p_1, n_1), \dots, \mathcal{Q}(p_k, n_k)$. Then $0 < \mu^*(U^*_1) < \infty$ and $\mu^*(U^*_t) = t^{2K^*} \mu^*(U^*_1)$, where $K^* = \sum_{j=1}^k (2p_j + n_j - 1)$ is half of the dimension of $\mathcal{Q}(p_1, n_1) \times \ldots \times \mathcal{Q}(p_k, n_k)$.

Obviously, $P^{(n)}(U; U_1, \dots, U_k) \subset P^{(n)}(U) \times U^*_1$. Suppose $\omega \in P^{(n)}(U)$ and $\omega^* \in U^*_1$. Then $(\omega, \omega^*) \in P^{(n)}_{\sigma,i}(U; U_1, \dots, U_k)$ if and only if $\omega \in P^{(n)}_{\sigma,i}(U)$ and $\omega^* \in tU^*_1$ for some $t > 0$ such that $a(\omega) + t^2 \le 1$, $a_i(\omega) > \sigma(a(\omega) + t^2)$. It follows that

$$\mu(P^{(n)}_{\sigma,i}(U; U_1, \dots, U_k)) = \mathfrak{m}(U) \mu^*(U^*_1) J_{n,K^*}(\sigma, i)$$

for all σ and i. Since $P^{(n)}_{0,i}(U; U_1, \dots, U_k) = P^{(n)}(U; U_1, \dots, U_k)$, statement (b) of the lemma follows from Lemma 8.1. ∎

Now we shall define 4 operations on translation surfaces and their equivalence classes: *cutting* along parallel saddle connections, *gluing* along parallel saddle connections, *collapsing* a short saddle connection, and *inserting* a short saddle connection. In what follows two saddle connections of a translation surface are called *homologous* if they are disjoint and break the surface into two parts. Note that homologous saddle connections are parallel and of the same length.

The cutting operation is defined on any translation surface X with a distinguished oriented saddle connection γ. If there is no saddle connection homologous to γ then the operation does nothing to X. Otherwise let $\gamma_1 = \gamma$ and $\gamma_2, \dots, \gamma_k$ be saddle connections homologous to γ. Orient $\gamma_2, \dots, \gamma_k$ so that they are of the same direction as γ. The surface X is divided by $\gamma_1, \dots, \gamma_k$ into k domains $C_1, \dots, C_k$, each domain being bounded by two saddle connections. We assume $\gamma_2, \dots, \gamma_k$ and $C_1, \dots, C_k$ are named so that each C_i is bounded by γ_i and γ_{i+1} (by definition, $\gamma_{k+1} = \gamma_1$) and, moreover, C_i lies to the right of γ_i (with respect to the direction of γ_i). Choose a partition τ of X by a maximal set of disjoint saddle connections containing $\gamma_1, \dots, \gamma_k$. As described in Section 5, the translation surface X can be obtained by gluing together plane triangles so that these triangles become cells of τ and edges of τ correspond to glued sides of the triangles. Reverse this construction and decompose X into disjoint plane triangles. For any i, $1 \le i \le k$, let T_i^+ denote the triangle corresponding to the cell of τ bounded by γ_i and contained in C_i. By T_i^- denote the triangle corresponding to the other cell of τ bounded by γ_i. Now glue the triangles in a different way. Namely, the side of T_i^+ parallel to γ is glued to a side of T_{i+1}^- (instead of T_i^-), where by definition $T_{k+1}^- = T_1^-$, while all other sides of triangles are glued together as before. The new gluing gives k distinct translation surfaces $X_1, \dots, X_k$. Namely, each X_i is obtained by gluing together triangles corresponding to cells of τ contained in C_i. By construction X_i is equipped with a triangulation τ_i by disjoint saddle connections. Also, we have a distinguished oriented edge γ_i' of τ_i that corresponds to glued sides of T_i^+ and T_{i+1}^-. X_i contains no saddle connection homologous to γ_i'. The sum of areas of $X_1, \dots, X_k$ is equal to the area of X.

Now suppose $X = M_p$ and the translation structure ω of X belongs to $\Omega(p, n)$ $(p, n \ge 1)$. Let $\tilde{\tau} \in \mathcal{T}(p, n)$ be the affine equivalence class of τ and $\tilde{\omega} \in N(\tilde{\tau})$ be the isotopy class of ω. Let $\tilde{\gamma}$ be the oriented edge of $\tilde{\tau}$ corresponding to γ (note that $\tilde{\gamma}$ is actually a homotopy class). Then the translation structure of any X_i $(1 \le i \le k)$ is isomorphic to some $\omega_i \in \Omega(p_i, n_i)$, where p_i and n_i are determined by $\tilde{\tau}$ and $\tilde{\gamma}$. The triangulation τ_i corresponds to a triangulation in a class $\tilde{\tau}_i \in \mathcal{T}(p_i, n_i)$. The class $\tilde{\tau}_i$ is determined up to the action of $\mathrm{Mod}(p_i, n_i)$ but once we fix it and specify an oriented edge $\tilde{\gamma}_i$ of $\tilde{\tau}_i$ corresponding to γ_i', the isotopy class $\tilde{\omega}_i \in N(\tilde{\tau}_i)$ of ω_i is uniquely determined by $\tilde{\omega}$, $\tilde{\tau}$, and $\tilde{\gamma}$. Thus we obtain a map $CUT[\tilde{\tau}, \tilde{\gamma}; \tilde{\tau}_1, \tilde{\gamma}_1, \dots, \tilde{\tau}_k, \tilde{\gamma}_k] : N(\tilde{\tau}) \to N(\tilde{\tau}_1) \times \ldots \times N(\tilde{\tau}_k)$, which is obviously affine.

The gluing operation is inverse to cutting. Let $X_1, \dots, X_k$ be translation surfaces and suppose each X_i has a distinguished oriented saddle connection γ_i. The gluing operation is defined if the holonomy vectors of $\gamma_1, \dots, \gamma_k$ coincide. For any X_i choose a triangulation τ_i by disjoint saddle connections including γ_i. Then decompose X_i into plane triangles according to the partition τ_i. We assume that all triangles obtained by decomposing surfaces $X_1, \dots, X_k$ are disjoint. Let T_i^+ denote the triangle that corresponds to the cell of τ_i bounded by γ_i and lying to the right of γ_i. By T_i^- denote the triangle corresponding to the other cell of τ_i bounded by γ_i. Now reglue the triangles in the following way. Let the side of T_i^+ parallel to γ_i be glued to the side of T_{i-1}^- parallel to γ_{i-1} (if $i = 1$, we assume $T_0^- = T_k^-$ and $\gamma_0 = \gamma_k$); this is possible as γ_i and γ_{i-1} have the same holonomy vector. All other sides of triangles are glued together as before. After the gluing we obtain a single translation surface X equipped with a triangulation τ by a maximal set of disjoint saddle connections. Let γ_i' $(1 \le i \le k)$ be the edge of τ that corresponds to glued sides of T_i^+ and T_{i-1}^-. Then $\gamma_1', \dots, \gamma_k'$ are homologous saddle connections of X. The surface X contains no more saddle connections homologous to $\gamma_1', \dots, \gamma_k'$ provided for any X_i there is no saddle connection homologous to γ_i. The area of X is the sum of areas of $X_1, \dots, X_k$.

Now suppose the translation structure ω_i of each X_i, $1 \le i \le k$, belongs to some $\Omega(p_i, n_i)$. Let $\tilde{\tau}_i \in \mathcal{T}(p_i, n_i)$ be the affine equivalence class of τ_i and $\tilde{\omega}_i \in N(\tilde{\tau}_i)$ be the isotopy class of ω_i. Let $\tilde{\gamma}_i$ be the oriented edge of $\tilde{\tau}_i$ corresponding to γ_i. Then the translation structure of X is isomorphic to some $\omega \in \Omega(p, n)$, where p and n are determined by $\tilde{\tau}_1, \dots, \tilde{\tau}_k$ and $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$. The triangulation τ corresponds to a triangulation in a class $\tilde{\tau} \in \mathcal{T}(p, n)$ that is determined up to the action of

$\mathrm{Mod}(p,n)$. Once we fix $\tilde{\tau}$ and specify its oriented edge $\tilde{\gamma}$ corresponding to γ_1', the isotopy class $\tilde{\omega} \in N(\tilde{\tau})$ of ω is uniquely determined by $\tilde{\omega}_i$, $\tilde{\tau}_i$, $\tilde{\gamma}_i$, $i = 1, \dots, k$. Thus we obtain a map $GLU[\tilde{\tau}_1, \tilde{\gamma}_1, \dots, \tilde{\tau}_k, \tilde{\gamma}_k; \tilde{\tau}, \tilde{\gamma}] : U \to N(\tilde{\tau})$ that is defined on the set U of $(\tilde{\omega}_1, \dots, \tilde{\omega}_k) \in N(\tilde{\tau}_1) \times \dots \times N(\tilde{\tau}_k)$ such that $\mathrm{hol}_{\tilde{\omega}_1}(\tilde{\gamma}_1) = \dots = \mathrm{hol}_{\tilde{\omega}_k}(\tilde{\gamma}_k)$. Note that U is an affine submanifold of $\mathcal{Q}(p_1, n_1) \times \dots \times \mathcal{Q}(p_k, n_k)$ and $GLU[\tilde{\tau}_1, \tilde{\gamma}_1, \dots, \tilde{\tau}_k, \tilde{\gamma}_k; \tilde{\tau}, \tilde{\gamma}]$ is an affine map.

We proceed to the collapsing operation. Let X be a translation surface with a distinguished oriented saddle connection γ. It is assumed that X has no saddle connection homologous to γ. Also, we require that X be not a torus with one singular point. Let τ be a triangulation of X by disjoint saddle connections including γ. By T^+ and T^- denote two triangles of τ bounded by γ. Assume that T^+ lies to the right of γ while T^- lies to the left. The saddle connection γ is collapsed as follows. We decompose X into plane triangles according to the partition τ and discard two triangles corresponding to T^+ and T^-. Then the remaining triangles are modified in a canonical way so that they can be glued together into another translation surface. In general, the collapsing operation is well-defined if γ is much shorter than the other edges of τ and angles of any triangle of τ not bounded by γ are not too small (precise conditions will be given later). Let $T_0 = T^+, T_1, \dots, T_k = T^-$ be a sequence of triangles of τ such that any two neighboring triangles have common edge different from γ and the length of the sequence is the least possible. By γ_i $(1 \le i \le k)$ denote a common edge of triangles T_{i-1} and T_i different from γ. Let γ_+ be the edge of T^+ different from γ and γ_1, and γ_- be the edge of T^- different from γ and γ_k. First consider the case when $k = 1$, that is, T^+ and T^- have two common edges γ and γ_1. Since X is not a torus with one singular point, $\gamma_+ \neq \gamma_-$. Clearly, γ_+ and γ_- are homologous saddle connections bounding a regular periodic cylinder. The collapsing operation in this case consists of removing the cylinder. Namely, after decomposing X and discarding two triangles only two sides of the remaining triangles cannot be glued as before. These sides correspond to homologous saddle connections γ_+ and γ_-, therefore they can be glued by translation. The other sides are glued as before and we obtain a translation surface X_0.

Now consider the case when $k > 1$, i.e., γ is the only common edge of T^+ and T^-. Then the edges γ_1, γ_+, γ_k, and γ_- are all distinct. Let T_+ be the triangle of τ bounded by γ_+ and distinct from T^+. Let T_- be the triangle bounded by γ_+ and distinct from T^-. Note that $T_+ \neq T_1$ as otherwise the edge of T_+ different from γ_+ and γ_1 would be homologous to γ. Similarly, $T_- \neq T_{k-1}$. Since the sequence $T_0, T_1, \dots, T_k$ is as short as possible, it does not contain triangles T_+ and T_-. Moreover, triangles $T_0, T_1, \dots, T_k$ are all distinct and so are saddle connections $\gamma_1, \dots, \gamma_k$. Decompose the translation surface X into disjoint plane triangles according to the triangulation τ. For any i, $1 \le i \le k-1$, let T_i' be the plane triangle corresponding to T_i. By γ_i' and γ_{i+1}'' denote the sides of T_i' corresponding to edges γ_i and γ_{i+1}, respectively. By A_i denote the common endpoint of γ_i' and γ_{i+1}''. Further, let T_+' and T_-' be the plane triangles corresponding to T_+ and T_-. Let γ_+' denote the side of T_+' corresponding to γ_+ and γ_-' denote the side of T_-' corresponding to γ_-. After discarding plane triangles corresponding to T^+ and T^-, the sides γ_1', γ_+', γ_k'', and γ_-' lose sides they were glued to. We are going to glue γ_1' to γ_+' and γ_k'' to γ_-' while the other sides of the remaining triangles will be glued as before. To make this possible, we shall modify triangles $T_1',\dots,T_{k-1}'$. Let v denote the holonomy vector of γ. The saddle connections γ_1 and γ_+ can be oriented so that their holonomy vectors differ by v or $-v$. It follows that we can move the vertex A_1 of the triangle T_1' by v or $-v$ so that sides γ_1' and γ_+' can be glued by translation. However this changes the side γ_1'' and it cannot be glued to γ_2' anymore. To fix this problem, we move the vertex A_2 of T_2' by v or $-v$, and so on. Finally each A_i is moved by v or $-v$. Note that each time the choice of a vector (v or $-v$) is uniquely determined by the configuration of triangles $T_0,T_1,\dots,T_k$ and saddle connections $\gamma,\gamma_+,\gamma_1,\dots,\gamma_k$. It is assumed that none of the triangles $T_1',\dots,T_{k-1}'$ degenerates while being modified, i.e., each A_i does not cross the straight line containing the opposite side of T_i'; otherwise the collapsing operation is not defined. Besides, we assume that all plane triangles are still disjoint after modifications. Then we can glue sides in each pair by translation and obtain a translation surface X_0. Indeed, this follows from the above for all pairs except γ_k'' and γ_-'. The sides γ_k'' and γ_-' can be glued by translation since the other pairs can.

Now suppose the translation structure ω of X belongs to some $\Omega(p,n)$. Let $\tilde\tau\in\mathcal{T}(p,n)$ be the affine equivalence class of τ and $\tilde\omega\in N(\tilde\tau)$ be the isotopy class of ω. Let $\tilde\gamma$ be the oriented edge of $\tilde\tau$ corresponding to γ. Then the translation structure of X_0 is isomorphic to some $\omega_0\in\Omega(p_0,n_0)$, where p_0 and n_0 are determined by $\tilde\tau$ and $\tilde\gamma$. By construction ω_0 is equipped with a triangulation τ_0 by disjoint saddle connections. We distinguish two edges of τ_0. In the case $k>1$, let e_r be the edge obtained by gluing the side γ_1' to γ_+' and e_l be the edge obtained by gluing γ_k'' to γ_-'. In the case $k=1$, let $e_r=e_l$ be the edge obtained by gluing together the sides corresponding to γ_+ and γ_-. We orient e_r and e_l so that pairs of vectors $(\mathrm{hol}_{\omega_0}(e_r),\mathrm{hol}_\omega(\gamma))$ and $(\mathrm{hol}_{\omega_0}(e_l),\mathrm{hol}_\omega(\gamma))$ induce the standard orientation in $\mathbb{R}^2$. Let $\tilde\tau_0\in\mathcal{T}(p_0,n_0)$ be the affine equivalence class of τ_0. $\tilde\tau_0$ is determined by $\tilde\tau$ and $\tilde\gamma$ up to the $\mathrm{Mod}(p_0,n_0)$ action on $\mathcal{T}(p_0,n_0)$. Let $\widetilde T_0,\dots,\widetilde T_k$ be triangles of $\tilde\tau$ corresponding to $T_0,\dots,T_k$ and $\tilde\gamma_1,\dots,\tilde\gamma_k$ be edges of $\tilde\tau$ corresponding to $\gamma_1,\dots,\gamma_k$. Note that $\widetilde T_0,\dots,\widetilde T_k$ and $\tilde\gamma,\tilde\gamma_1,\dots,\tilde\gamma_k$ are defined up to the $H_0(p,n)$ action on M_p. Once we fix $\tilde\tau_0$ and specify its oriented edges $\tilde e_r$, $\tilde e_l$ corresponding to e_r, e_l, the isotopy class $\tilde\omega_0$ of ω_0 is uniquely determined by $\tilde\omega,\tilde\tau,\tilde\gamma,\widetilde T_0,\dots,\widetilde T_k,\tilde\gamma_1,\dots,\tilde\gamma_k$. Thus we obtain a map $COL[\tilde\tau,\tilde\gamma;\widetilde T_0,\dots,\widetilde T_k;\tilde\gamma_1,\dots,\tilde\gamma_k;\tilde\tau_0,\tilde e_r,\tilde e_l]:U\to N(\tilde\tau_0)$, where $U\subset N(\tilde\tau)$. This map is affine, namely, there exists a linear mapping $f:H^1(M_p,Z_n;\mathbb{R}^2)\to H^1(M_{p_0},Z_{n_0};\mathbb{R}^2)$ such that $\mathrm{dev}(COL[\tilde\tau,\tilde\gamma;\widetilde T_0,\dots,\widetilde T_k;\tilde\gamma_1,\dots,\tilde\gamma_k;\tilde\tau_0,\tilde e_r,\tilde e_l](\tilde\omega))=f(\mathrm{dev}(\tilde\omega))$ for any $\tilde\omega\in U$. It follows from the construction that U is the set of $\tilde\omega\in N(\tilde\tau)$ such that $f(\mathrm{dev}(\tilde\omega))\in\mathrm{dev}(N(\tilde\tau_0))$ (cf. the proof of Lemma 6.1).

The inserting operation is inverse to collapsing. Let X_0 be a translation surface and τ_0 be a triangulation of X_0 by disjoint saddle connections. Let e_r, e_l be oriented edges of τ_0 and v_r, v_l be their holonomy vectors. Finally, let v be a nonzero vector in $\mathbb{R}^2$ such that pairs of vectors (v_r,v) and (v_l,v) induce the standard orientation in $\mathbb{R}^2$. The inserting operation results in a translation surface X that has a saddle connection with holonomy vector v. In general, the operation is defined if v is short enough (precise conditions will be given later). First consider the case when $e_l=e_r$.

Let T_+ and T_- be triangles of τ_0 bounded by e_r. We assume that the orientation of e_r agrees with the counterclockwise orientation of the boundary of T_+. Decompose X_0 into disjoint plane triangles according to the partition τ_0. Let T'_+ and T'_- be the plane triangles corresponding to T_+ and T_-. Let γ'_+ and γ'_- denote the sides of T'_+ and T'_-, respectively, corresponding to e_r. Add a parallelogram $B_1B_2B_3B_4$ such that $B_1 - B_2 = B_4 - B_3 = v$, $B_3 - B_2 = B_4 - B_1 = v_r$. Now glue all triangles and the parallelogram together as follows. The side B_2B_1 of the parallelogram is glued to its side B_3B_4. The sides B_1B_4 and B_2B_3 are glued to γ'_+ and γ'_-, respectively. All the other sides are glued as before. We obtain a translation surface X.

Now consider the case $e_l \neq e_r$. Choose a sequence $T_1, \dots, T_k$ of triangles of τ_0 such that e_r is an edge of T_1, e_l is an edge of T_k, any two neighboring triangles have common edge different from e_r and e_l, and the length of the sequence is the least possible. By γ_i ($1 \leq i \leq k-1$) denote a common edge of triangles T_i and T_{i+1} different from e_r and e_l. Also, let $\gamma_0 = e_r$ and $\gamma_k = e_l$. Let T_+ be the triangle of τ_0 bounded by e_r and distinct from T_1. Let T_- be the triangle bounded by e_l and distinct from T_k. Since the sequence $T_1, \dots, T_k$ is as short as possible, it does not contain T_+ and T_-. Also, triangles $T_1, \dots, T_k$ are all distinct and so are saddle connections $\gamma_1, \dots, \gamma_{k-1}$. Decompose X_0 into disjoint plane triangles according to the partition τ_0. For any i, $1 \leq i \leq k$, let T'_i be the plane triangle corresponding to T_i. By γ'_{i-1} and γ''_i denote the sides of T'_i corresponding to edges γ_{i-1} and γ_i, respectively. By A_i denote the common endpoint of γ'_{i-1} and γ''_i. Further, let T'_+ and T'_- be the plane triangles corresponding to T_+ and T_-. Let γ'_+ denote the side of T'_+ corresponding to e_r and γ'_- denote the side of T'_- corresponding to e_l. We add two more plane triangles T'_r and T'_l with the following properties. First of all, T'_r and T'_l have oriented sides γ'_r and γ'_l equal to v as vectors. Moreover, T'_r lies to the right of γ'_r while T'_l lies to the left of γ'_l. Further, T'_r has a side e'_+ that can be glued to the side γ'_+ of T'_+ while T'_l has a side e'_- that can be glued to the side γ'_- of T'_-. By e'_r denote the side of T'_r different from γ'_r and e'_+. By e'_l denote the side of T'_l different from γ'_l and e'_-. Now we are going to glue γ'_r to γ'_l, e'_+ to γ'_+, e'_- to γ'_-, e'_r to γ'_0, e'_l to γ''_k, while the other sides of the triangles will be glued as before. At this point the sides e'_r and γ'_0 cannot be glued by translation but we can move the vertex A_1 of the triangle T'_1 by v or $-v$ so that the gluing is possible. However this changes the side γ''_1 and it cannot be glued to γ'_1 anymore. To fix this problem, we move the vertex A_2 of T'_2 by v or $-v$, and so on. Finally each A_i is moved by v or $-v$, where the choice of a vector is uniquely determined by the configuration of triangles $T_1, \dots, T_k$ and saddle connections $\gamma_0, \gamma_1, \dots, \gamma_k$. It is assumed that all plane triangles remain disjoint after modifications. The inserting operation is defined if none of the triangles $T'_1, \dots, T'_k$ degenerates while being modified. Then we can glue sides in each pair by translation and obtain a translation surface X. Indeed, this follows from the above for all pairs except e'_l and γ''_k. The sides e'_l and γ''_k can be glued by translation since the other pairs can.

Now suppose the translation structure ω_0 of X_0 belongs to some $\Omega(p_0, n_0)$. Let $\tilde{\tau}_0 \in \mathcal{T}(p_0, n_0)$ be the affine equivalence class of τ_0 and $\tilde{\omega}_0 \in N(\tilde{\tau}_0)$ be the isotopy class of ω_0. Let $\tilde{e}_r$ and $\tilde{e}_l$ be oriented edges of $\tilde{\tau}_0$ corresponding to e_r and e_l. Then the translation structure of X is isomorphic to some $\omega \in \Omega(p,n)$, where p and n depend on $\tilde{\tau}_0$, $\tilde{e}_l$, and $\tilde{e}_r$. In both cases, $e_l = e_r$ and $e_l \neq e_r$, the surface X is equipped with a partition τ by disjoint saddle connections. Moreover, τ has a distinguished oriented edge γ with holonomy vector v. If $e_l \neq e_r$ then τ is a triangulation in a class $\tilde{\tau} \in \mathcal{T}(p,n)$ that is determined by $\tilde{\tau}_0$, $\tilde{e}_l$, and $\tilde{e}_r$ up to the $\mathrm{Mod}(p,n)$ action. If $e_l = e_r$ then one of cells of τ is a quadrilateral. We can make τ into a triangulation by adding a diagonal of the quadrilateral. Two triangulations obtained this way are not affine equivalent but they correspond to the same element of $\mathcal{MT}(p,n)$. Hence the affine equivalence class $\tilde{\tau}$ of any of them is determined up to the action of $\mathrm{Mod}(p,n)$. Let us fix $\tilde{\tau}$ and specify its oriented edge $\tilde{\gamma}$ corresponding to γ. In the case $e_l = e_r$ the isotopy class $\tilde{\omega} \in N(\tilde{\tau})$ of ω is then uniquely determined by $v, \tilde{\omega}_0, \tilde{\tau}_0$, and $\tilde{e}_r = \tilde{e}_l$. In the case $e_l \neq e_r$, let $\widetilde{T}_1, \dots, \widetilde{T}_k$ be triangles of $\tilde{\tau}_0$ corresponding to $T_1, \dots, T_k$ and $\tilde{\gamma}_1, \dots, \tilde{\gamma}_{k-1}$ be edges of $\tilde{\tau}_0$ corresponding to $\gamma_1, \dots, \gamma_{k-1}$. Then $\tilde{\omega}$ is uniquely determined by $v, \tilde{\omega}_0, \tilde{\tau}_0, \tilde{e}_r, \tilde{e}_l, \widetilde{T}_1, \dots, \widetilde{T}_k, \tilde{\gamma}_1, \dots, \tilde{\gamma}_{k-1}$. In any case we get a map $INS[\tilde{\tau}_0, \tilde{e}_r, \tilde{e}_l; \widetilde{T}_1, \dots, \widetilde{T}_k; \tilde{\gamma}_1, \dots, \tilde{\gamma}_{k-1}; \tilde{\tau}, \tilde{\gamma}] : U_0 \to N(\tilde{\tau})$ defined on a set $U_0 \subset \mathbb{R}^2 \times N(\tilde{\tau}_0)$. The map $INS[\mathcal{D}]$ (here $\mathcal{D}$ stands for the set of parameters of the operation) is affine, namely, there exists a linear mapping $f_0 : \mathbb{R}^2 \times H^1(M_{p_0}, Z_{n_0}; \mathbb{R}^2) \to H^1(M_p, Z_n; \mathbb{R}^2)$ such that $\mathrm{dev}(INS[\mathcal{D}](v, \tilde{\omega}_0)) = f_0(v, \mathrm{dev}(\tilde{\omega}_0))$ for any $(v, \tilde{\omega}_0) \in U_0$. It follows from the construction that U_0 is the set of $(v, \tilde{\omega}_0) \in \mathbb{R}^2 \times N(\tilde{\tau}_0)$ such that $f_0(v, \mathrm{dev}(\tilde{\omega}_0)) \in \mathrm{dev}(N(\tilde{\tau}))$.

Now we define more complex operation on translation surfaces. Suppose X_0 is a translation torus with n singular points such that the translation structure ω_0 of X_0 belongs to $\Omega(1,n)$ and the isotopy class $\tilde{\omega}_0$ of ω_0 belongs to $P^{(n)} \subset N(\tau^{(n)})$. Further, let $X_1, \dots, X_k$ be translation surfaces. The triangulation in the class $\tau^{(n)}$ of X_0 has distinguished oriented edge L_1 in the homotopy class $L^{(n)}$. By v denote the holonomy vector of L_1. First the cutting operation is applied to the surface X_0 and the edge L_1. We yield an ordered sequence of n translation tori $X_{01}, \dots, X_{0n}$. Further for any X_j, $1 \leq j \leq k$, we insert a saddle connection with holonomy vector v by applying an inserting operation. Assuming the inserting operation is well defined, we get a translation surface X'_j. Suppose $l_1, \dots, l_k$ are integers such that $0 \leq l_1 \leq \ldots \leq l_k \leq n$. We merge two sequences $X_{01}, \dots, X_{0n}$ and $X'_1, \dots, X'_k$ together in such a way that X_{0i} appears before X'_j if and only if $i \leq l_j$. Each surface in the new sequence has a distinguished saddle connection with holonomy vector v. Finally we apply the gluing operation and obtain a translation surface X.

Now suppose the translation structure ω_j of every X_j, $1 \leq j \leq k$, belongs to some $\Omega(p_j, n_j)$. Let $\omega, \omega_{01}, \dots, \omega_{0n}, \omega'_1, \dots, \omega'_k$ denote the translation structures of $X, X_{01}, \dots, X_{0n}, X'_1, \dots, X'_k$, respectively. It is no loss to assume that each of these translation structures also belongs to some $\Omega(p', n')$. Let the tilde denote the isotopy class of a translation structure. Then $(\tilde{\omega}_{01}, \dots, \tilde{\omega}_{0n}) = CUT[\tau^{(n)}, L^{(n)}; \mathcal{D}_0](\tilde{\omega}_0)$ for a set $\mathcal{D}_0$ of parameters. Further, $\tilde{\omega}'_j = INS[\tau_j, \mathcal{D}_j; \tau'_j, \gamma'_j](v, \tilde{\omega}_j)$, where $\tau_j \in \mathcal{T}(p_j, n_j)$ and $\mathcal{D}_j$ is a set of parameters. Finally,

$$\tilde{\omega} = GLU[\mathcal{D}'; \tau, \gamma](\tilde{\omega}_{01}, \dots, \tilde{\omega}_{0l_1}, \tilde{\omega}'_1, \dots, \tilde{\omega}'_k, \tilde{\omega}_{0,l_k+1}, \dots, \tilde{\omega}_{0n})$$

for another set $\mathcal{D}'$ of parameters. Thus we yield a map

$$OPR[n;\tau_1,\mathcal{D}_1,\dots,\tau_k,\mathcal{D}_k;l_1,\dots,l_k;\tau,\gamma] : U^* \to N(\tau)$$

defined on an open set $U^* \subset P^{(n)} \times N(\tau_1) \times \ldots \times N(\tau_k)$. We shall use the notation $OPR[n;\mathcal{D}]$, that is, $\mathcal{D}$ stands for all the other parameters. Suppose $\tilde{\omega}_0 \in P^{(n)}$ and $\tilde{\omega}_j \in N(\tau_j)$, $j = 1,\dots,k$. Then $(\tilde{\omega}_0,\tilde{\omega}_1,\dots,\tilde{\omega}_k) \in U^*$ if and only if $INS[\tau_j,\mathcal{D}_j;\tau_j',\gamma_j'](\mathrm{hol}_{\tilde{\omega}_0}(L^{(n)}),\tilde{\omega}_j)$ is well defined for $1 \le j \le k$.

Lemma 8.3 *The map $OPR[n;\mathcal{D}]$ is affine, injective, and volume preserving.*

Proof. Suppose $\omega = OPR[n;\mathcal{D}](\omega_0,\omega_1,\dots,\omega_k)$. By construction every triangle of triangulations $\tau^{(n)},\tau_1,\dots,\tau_k$ is assigned a triangle of τ while every edge is assigned one or two edges of τ. Let γ be an oriented edge of $\tau^{(n)}$ or τ_j, $1 \le j \le k$, and γ' be a corresponding edge of τ. If γ is an edge of $\tau^{(n)}$ then $\mathrm{hol}_\omega(\gamma') = \mathrm{hol}_{\omega_0}(\gamma)$. If γ is an edge of τ_j then $\mathrm{hol}_\omega(\gamma') = \mathrm{hol}_{\omega_j}(\gamma) + c_\gamma \mathrm{hol}_{\omega_0}(L^{(n)})$, where $c_\gamma \in \{-1,0,1\}$ depends only on γ.

Let $\gamma_{01} = L^{(n)},\gamma_{02},\dots,\gamma_{0s_0}$ be a maximal set of edges of $\tau^{(n)}$ that do not divide the surface. For any j, $1 \le j \le k$, let $\gamma_{j1},\dots,\gamma_{js_j}$ be a maximal set of edges of τ_j that do not divide the surface. Let γ_{ji}' denote an edge of τ assigned to γ_{ji}. It is easy to observe that the edges γ_{ji}', $1 \le i \le s_j$, $0 \le j \le k$, are all distinct and comprise a maximal set of edges of τ that do not divide the surface. Let $K = s_0 + s_1 + \dots + s_k$. Define maps $f_1 : N(\tau^{(n)}) \times N(\tau_1) \times \ldots \times N(\tau_k) \to (\mathbb{R}^2)^K$ and $f_2 : N(\tau) \to (\mathbb{R}^2)^K$ by

$$f_1(\omega_0,\omega_1,\dots,\omega_k) = (\mathrm{hol}_{\omega_0}(\gamma_{01}),\dots,\mathrm{hol}_{\omega_0}(\gamma_{0s_0}),\mathrm{hol}_{\omega_1}(\gamma_{11}),\dots,\mathrm{hol}_{\omega_k}(\gamma_{ks_k})),$$

$$f_2(\omega) = (\mathrm{hol}_\omega(\gamma_{01}'),\dots,\mathrm{hol}_\omega(\gamma_{0s_0}'),\mathrm{hol}_\omega(\gamma_{11}'),\dots,\mathrm{hol}_\omega(\gamma_{ks_k}')).$$

Both maps are affine. By Lemmas 6.1 and 6.2, f_1 and f_2 are injective and volume preserving. By the above there exists a linear map $F : (\mathbb{R}^2)^K \to (\mathbb{R}^2)^K$ such that $f_2(OPR[n;\mathcal{D}](\omega^*)) = F(f_1(\omega^*))$ for any $\omega^* \in U^*$. Given $v_1,\dots,v_K \in \mathbb{R}^2$, one has $F(v_1,v_2,\dots,v_K) = (v_1,v_2+c_2v_1,\dots,v_K+c_Kv_1)$, where $c_2,\dots,c_K \in \{-1,0,1\}$ are constants. It follows that F is volume preserving. Then $OPR[n;\mathcal{D}]$ is a volume preserving affine map. It is injective since f_1 and F are injective. ∎

Suppose $U_0 \subset \mathbb{R}^2$ and $U_j \subset N(\tau_j) \cap \mathcal{Q}_1(p_j,n_j)$, $j = 1,\dots,k$. Consider the set $P^{(n)}[\mathcal{D}](U_0;U_1,\dots,U_k)$ consisting of $\omega \in N(\tau)$ such that $a(\omega) \le 1$ and $\omega = OPR[n;\mathcal{D}](\omega_0,t_1\omega_1,\dots,t_k\omega_k)$, where $\omega_0 \in P^{(n)}(U_0)$, $\omega_j \in U_j$ and $t_j > 0$ for $1 \le j \le k$. Given $i \in \{1,\dots,n\}$ and $\sigma \in [0,1)$, we let $\omega \in P^{(n)}_{\sigma,i}[\mathcal{D}](U_0;U_1,\dots,U_k)$ if, in addition, $a_i(\omega_0) > \sigma a(\omega)$.

Lemma 8.4 *Let $U_0 \subset \mathbb{R}^2 \setminus \{(0,0)\}$ and $U_j \subset N(\tau_j) \cap \mathcal{Q}_1(p_j,n_j)$, $1 \le j \le k$, be nonempty open subsets. Assume U_0 is bounded and the closure of each U_j, $1 \le j \le k$, is a compact subset of $N(\tau_j)$. Further assume that for any $v \in U_0$ and $\omega_j \in U_j$ ($1 \le j \le k$) the j^{th} inserting operation is defined on (tv,ω_j) when $t > 0$ is small enough. Then (a)*

$$\lim_{\epsilon\to 0} \epsilon^{-2}\mu(P^{(n)}[\mathcal{D}](\epsilon U_0;U_1,\dots,U_k)) = \mu^*(P^{(n)}(U_0;U_1,\dots,U_k)) > 0,$$

where μ is the canonical measure on $N(\tau)$ and μ^* denotes the product of the canonical measures on $\mathcal{Q}(1,n), \mathcal{Q}(p_1,n_1), \dots, \mathcal{Q}(p_k,n_k)$;

(b) for any $\sigma \in [0,1)$ and $i \in \{1,\dots,n\}$,

$$\lim_{\epsilon\to 0} \frac{\mu(P^{(n)}_{\sigma,i}[\mathcal{D}](\epsilon U_0; U_1,\dots,U_k))}{\mu(P^{(n)}[\mathcal{D}](\epsilon U_0; U_1,\dots,U_k))} = (1-\sigma)^{K-2},$$

where K is half of the dimension of $N(\tau)$;

(c) if, in addition, $U_0, U_1, \dots, U_k$ are connected sets and $tU_0 \subset U_0$ for $0<t<1$, then the set $P^{(n)}[\mathcal{D}](U_0; U_1,\dots,U_k)$ is connected.

Proof. Consider an inserting operation $f_0 = INS[\tau_0, e_r, e_l; \mathcal{D}_0; \tau', \gamma']$. Here $\mathcal{D}_0$ is empty if $e_r = e_l$; otherwise $\mathcal{D}_0$ stands for a sequence $T_1,\dots,T_s$ of triangles and a sequence $\gamma_1,\dots,\gamma_{s-1}$ of edges of τ_0. The operation is defined on a set $U_0^* \subset \mathbb{R}^2 \times N(\tau_0)$. Suppose $v \in \mathbb{R}^2$ and $\omega \in N(\tau_0)$. Let Λ denote the euclidean area form on $\mathbb{R}^2$. If $(v,\omega) \in U_0^*$, then $\Lambda(\mathrm{hol}_\omega(e_r), v) > 0$ and $\Lambda(\mathrm{hol}_\omega(e_l), v) > 0$. In the case $e_r = e_l$, these conditions determine the set U_0^*. In the case $e_r \neq e_l$, there are also s nondegeneracy conditions. They can be expressed in the form $\Lambda(\mathrm{hol}_\omega(L_{1i}) + v, \mathrm{hol}_\omega(L_{2i})) > 0$, $i = 1,\dots,s$, where L_{1i}, L_{2i} are certain oriented edges of τ_0 such that $\Lambda(\mathrm{hol}_\omega(L_{1i}), \mathrm{hol}_\omega(L_{2i})) > 0$ for any $\omega \in N(\tau_0)$. Namely, L_{1i} and L_{2i} are edges of the triangle T_i such that L_{1i} is in the sequence $e_r, \gamma_1, \dots, \gamma_{s-1}, e_l$ while L_{2i} is not. It follows that U_0^* is an open set. Furthermore, if $(v,\omega) \in U_0^*$ then $(tv, t\omega) \in U_0^*$ for any $t>0$ and $(tv,\omega) \in U_0^*$ for $0<t<1$. If $\Lambda(\mathrm{hol}_\omega(e_r), v) > 0$ and $\Lambda(\mathrm{hol}_\omega(e_l), v) > 0$ then $(tv,\omega) \in U_0^*$ provided $t>0$ is small enough. Each triangle T of τ_0 corresponds to a triangle T' of τ'. Suppose $(v,\omega) \in U_0^*$ and $\omega' = f_0(v,\omega)$. If T is not in the sequence $T_1,\dots,T_s$ then the area of T with respect to ω equals the area of T' with respect to ω'. Otherwise the two areas may differ but the difference is at most $C_\omega |v|/2$, where C_ω is the maximum length of edges of τ_0 with respect to ω. Besides, there are two triangles of τ' bounded by γ' that are not associated to triangles of τ_0. Their areas with respect to ω' do not exceed $C_\omega|v|/2$. It follows that $|a(\omega') - a(\omega)| \le m_0 C_\omega |v|$, where m_0 is half of the number of triangles of τ'.

Now consider the map $f = OPR[n;\mathcal{D}]$ defined on the set $U^* \subset P^{(n)} \times N(\tau_1) \times \dots \times N(\tau_k)$. Suppose $(\omega_0, \omega_1, \dots, \omega_k) \in U^*$. By the above $(t\omega_0, t\omega_1, \dots, t\omega_k) \in U^*$ for any $t>0$, $(t\omega_0, \omega_1,\dots,\omega_k) \in U^*$ for $0<t<1$, and $(\omega_0, t_1\omega_1, \dots, t_k\omega_k) \in U^*$ for $t_1,\dots,t_k \ge 1$. The set U_0 is bounded, i.e., it is contained in the disk $B(C_0)$ for some $C_0 > 0$. Since the closure of each U_j, $1 \le j \le k$, is a compact subset of $N(\tau_j)$, U_j has finite volume in $\mathcal{Q}_1(p_j, n_j)$. Also, there exist $C, c, \delta > 0$ such that for any triangulation in the class τ_j of a translation surface in an isotopy class $\omega_j \in U_j$ all edges are of length at most C and at least c while all angles of each triangle are not less than δ. Since for any $v \in U_0$ and $\omega_j \in U_j$ the j^{th} inserting operation is defined on (tv, ω_j) when $t>0$ is small enough, there exists a constant $\epsilon_0 > 0$ depending on C_0, c, δ such that the j^{th} inserting operation is defined on $(\epsilon_0 v, \omega_j)$. It follows that $P^{(n)}(\epsilon_0 U_0) \times U_1 \times \dots \times U_k \subset U^*$.

For any $(\omega_0, \omega_1, \dots, \omega_k) \in P^{(n)} \times N(\tau_1) \times \dots \times N(\tau_k)$ let $a^*(\omega_0, \omega_1, \dots, \omega_k) = a(\omega_0) + a(\omega_1) + \dots + a(\omega_k)$. Suppose $\omega = f(\omega^*)$ for some $\omega^* = (\omega_0, t_1\omega_1, \dots, t_k\omega_k)$, where $\omega_0 \in P^{(n)}(\epsilon U_0)$, $\epsilon > 0$, $\omega_j \in U_j$ and $t_j > 0$ for $1 \le j \le k$. By the above $|a^*(\omega^*) - a(\omega)| \le mCt_{\max}|\operatorname{hol}_{\omega_0}(L^{(n)})|$, where $t_{\max} = \max(t_1, \dots, t_k)$ and m is half of the number of triangles of the triangulation τ. Moreover, $|\operatorname{hol}_{\omega_0}(L^{(n)})| \le C_0\epsilon$. If $a^*(\omega^*) \le 1$ then $t_{\max} \le 1$. If $a(\omega) \le 1$ then $|a^*(\omega^*) - a(\omega)| > t_{\max}^2 - 1$; it follows that $t_{\max} < 2$ whenever $\epsilon \le 3(2mC_0C)^{-1}$. Hence there exist $C_1, \epsilon_1 > 0$ such that $|a^*(\omega^*) - a(\omega)| \le C_1\epsilon$ if $a^*(\omega^*) \le 1$ or $a(\omega) \le 1$ and $\epsilon \le \epsilon_1$.

For any $\epsilon, \alpha > 0$ let $Y(\epsilon, \alpha)$ be the set of $(\omega_0, \dots, \omega_k) \in P^{(n)} \times \mathbb{R}^+U_1 \times \dots \times \mathbb{R}^+U_k$ such that $\operatorname{hol}_{\omega_0}(L^{(n)}) \in \epsilon U_0$ and $a^*(\omega_0, \dots, \omega_k) \le \alpha$. Observe that $tY(\epsilon, \alpha) = Y(t\epsilon, t^2\alpha)$ for any $t > 0$, where by definition $t(\omega_0, \dots, \omega_k) = (t\omega_0, \dots, t\omega_k)$. Clearly, $Y(\epsilon, 1) = P^{(n)}(\epsilon U_0; U_1, \dots, U_k)$. It was shown in the proof of Lemma 8.2 that $\mu^*(P^{(n)}(\epsilon U_0; U_1, \dots, U_k)) = \mathfrak{m}(\epsilon U_0)c^*$, where $c^* > 0$ depends on $U_1, \dots, U_k$, and n. Hence $\mu^*(Y(\epsilon, \alpha)) = \alpha^K \mu^*(Y(\alpha^{-1/2}\epsilon, 1)) = \alpha^{K-1}\epsilon^2\mathfrak{m}(U_0)c^*$.

For any $\epsilon > 0$ let $Y_0(\epsilon) = f^{-1}(P^{(n)}[\mathcal{D}](\epsilon U_0; U_1, \dots, U_k)) \cap P^{(n)}(\epsilon U_0; U_1, \dots, U_k)$, $Y_1(\epsilon) = f^{-1}(P^{(n)}[\mathcal{D}](\epsilon U_0; U_1, \dots, U_k)) \setminus Y_0(\epsilon)$, $Y_2(\epsilon) = P^{(n)}(\epsilon U_0; U_1, \dots, U_k) \setminus Y_0(\epsilon)$, $Y_3(\epsilon) = Y_2(\epsilon) \cap U^*$, $Y_4(\epsilon) = Y_2(\epsilon) \setminus U^*$. If $\omega^* \in Y_1(\epsilon)$ then $a^*(\omega^*) > 1$ and $a(f(\omega^*)) \le 1$. If $\omega^* \in Y_3(\epsilon)$ then $a(f(\omega^*)) > 1$ and $a^*(\omega^*) \le 1$. In both cases, $|a^*(\omega^*) - 1| \le C_1\epsilon$ provided $\epsilon \le \epsilon_1$. Therefore $Y_1(\epsilon) \cup Y_3(\epsilon)$ is a subset of $Y(\epsilon, 1 + C_1\epsilon)$ while it is disjoint from $Y(\epsilon, \alpha)$ for $\alpha < 1 - C_1\epsilon$. It follows that $\mu^*(Y_1(\epsilon) \cup Y_3(\epsilon)) \le ((1 + C_1\epsilon)^{K-1} - (1 - C_1\epsilon)^{K-1})\epsilon^2\mathfrak{m}(U_0)c^*$ if $C_1\epsilon < 1$. Hence $\epsilon^{-2}\mu^*(Y_1(\epsilon) \cup Y_3(\epsilon)) \to 0$ as $\epsilon \to 0$.

It is easy to see that $P^{(n)}(\epsilon U_0; U_1, \dots, U_k) \subset P^{(n)}(\epsilon U_0) \times \widetilde{U}_1 \times \dots \times \widetilde{U}_k$, where $\widetilde{U}_j = \{t\omega \mid \omega \in U_j,\ 0 < t \le 1\}$. Let $\mu_0, \mu_1, \dots, \mu_k$ be the canonical measures on $\mathcal{Q}(1, n), \mathcal{Q}(p_1, n_1), \dots, \mathcal{Q}(p_k, n_k)$, respectively. Then $\mu_0(P^{(n)}(\epsilon U_0)) = \mathfrak{m}(\epsilon U_0)c_n = \epsilon^2\mathfrak{m}(U_0)c_n$, where $c_n > 0$ is a constant (see the proof of Lemma 8.2). Besides, $\mu_j(\widetilde{U}_j) < \infty$. Suppose $(\omega_0, \dots, \omega_k) \in Y_4(\epsilon)$. Then $(\omega_0, \dots, \omega_k) \notin U^*$, hence $\omega_j \in \epsilon\epsilon_0^{-1}\widetilde{U}_j$ for some $1 \le j \le k$. Since $\mu_j(\epsilon\epsilon_0^{-1}\widetilde{U}_j) = (\epsilon/\epsilon_0)^{2(2p_j+n_j-1)}\mu_j(\widetilde{U}_j)$, it follows that $\epsilon^{-2}\mu^*(Y_4(\epsilon)) \to 0$ as $\epsilon \to 0$. Thus we have proved that $\epsilon^{-2}\mu^*(Y_1(\epsilon) \cup Y_2(\epsilon)) \to 0$ as $\epsilon \to 0$. Then $\lim_{\epsilon\to 0} \epsilon^{-2}\mu^*(Y_0(\epsilon)) = \mathfrak{m}(U_0)c^* > 0$. Since f is injective and volume preserving map, statement (a) of the lemma follows.

Suppose $\omega^* \in Y_0(\epsilon) \setminus Y(\epsilon, C_1\epsilon^{1/2})$, where $\epsilon \le \epsilon_1$. Then $|a(f(\omega^*)) - a^*(\omega^*)| \le C_1\epsilon \le \epsilon^{1/2}a^*(\omega^*)$. If $f(\omega^*) \in P_{\sigma,i}^{(n)}[\mathcal{D}](\epsilon U_0; U_1, \dots, U_k)$ for some $\sigma > 0$ and i, then $\omega^* \in P_{(1-\epsilon^{1/2})\sigma,i}^{(n)}(\epsilon U_0; U_1, \dots, U_k)$ assuming $\epsilon < 1$, else $\omega^* \notin P_{(1+\epsilon^{1/2})\sigma,i}^{(n)}(\epsilon U_0; U_1, \dots, U_k)$ assuming $(1 + \epsilon^{1/2})\sigma < 1$. Therefore

$$\mu^*(P_{(1+\epsilon^{1/2})\sigma,i}^{(n)}(\epsilon U_0; U_1, \dots, U_k)) - \mu^*(Y(\epsilon, C_1\epsilon^{1/2}) \cup Y_2(\epsilon)) \le$$

$$\mu(P_{\sigma,i}^{(n)}[\mathcal{D}](\epsilon U_0; U_1, \dots, U_k)) \le$$

$$\mu^*(P_{(1-\epsilon^{1/2})\sigma,i}^{(n)}(\epsilon U_0; U_1, \dots, U_k)) + \mu^*(Y(\epsilon, C_1\epsilon^{1/2}) \cup Y_1(\epsilon)).$$

Since $\epsilon^{-2}\mu^*(Y(\epsilon, C_1\epsilon^{1/2}) \cup Y_1(\epsilon) \cup Y_2(\epsilon)) \to 0$ as $\epsilon \to 0$, statement (b) of the lemma follows from its statement (a) and Lemma 8.2.

Now assume $U_0, U_1, \dots, U_k$ are connected sets. It is easy to observe that the set $P^{(n)}(\{v\})$ is connected for any nonzero $v \in \mathbb{R}^2$. Let g_v be a unique element of $\mathrm{SL}(2, \mathbb{R})$ such that $g_v v = (0, 1)$ and $g_v v_0 = (1, 0)$ for some v_0 orthogonal to

v. Then $g_v P^{(n)}(\{v\}) = P^{(n)}(\{(0,1)\})$. Since g_v depends continuously on v, it follows that $P^{(n)}(U_0)$ is connected whenever $U_0 \subset \mathbb{R}^2 \setminus \{(0,0)\}$ is connected. Suppose $\omega', \omega'' \in P^{(n)}[\mathcal{D}](U_0; U_1, \dots, U_k)$. Then $f^{-1}(\omega') = (\omega'_0, t'_1\omega'_1, \dots, t'_k\omega'_k)$ and $f^{-1}(\omega'') = (\omega''_0, t''_1\omega''_1, \dots, t''_k\omega''_k)$, where $\omega'_0, \omega''_0 \in P^{(n)}(U_0)$, $\omega'_j, \omega''_j \in U_j$ and $t'_j, t''_j > 0$ for $1 \leq j \leq k$. There exist continuous paths $\omega_0 : [0,1] \to P^{(n)}(U_0)$ and $\omega_j : [0,1] \to U_j$, $j = 1, \dots, k$, such that $\omega_j(0) = \omega'_j$ and $\omega_j(1) = \omega''_j$ for $0 \leq j \leq k$. Pick $t_0 > 0$ such that $t'_j, t''_j \geq t_0$ and $\epsilon_0 t_0 \leq 1$. For any $u \in [0,1]$ let

$$\omega^*(u) = (\epsilon_0 t_0 \omega_0(u), ((1-u)t'_1 + ut''_1)\omega_1(u), \dots, ((1-u)t'_k + ut''_k)\omega_k(u)).$$

Then $\omega^*(u) \in U^*$ and depends continuously on u. In particular, $u \mapsto a(f(\omega^*(u)))$ is a continuous function on $[0,1]$, hence it is bounded. Note that $a(f(t\omega^*(u))) = a(tf(\omega^*(u))) = t^2 a(f(\omega^*(u)))$ for any $t > 0$. Therefore $a(f(t_1\omega^*(u))) \leq 1$ for some $0 < t_1 < 1$ and all u. Assume $tU_0 \subset U_0$ for $0 < t < 1$. Then $f(t_1\omega^*(u)) \in P^{(n)}[\mathcal{D}](U_0; U_1, \dots, U_k)$ for all $u \in [0,1]$, hence $f(t_1\omega^*(0))$ is joined to $f(t_1\omega^*(1))$ by a continuous path in $P^{(n)}[\mathcal{D}](U_0; U_1, \dots, U_k)$. Furthermore, $P^{(n)}[\mathcal{D}](U_0; U_1, \dots, U_k)$ contains $f(t\omega^*(0))$ for $t_1 \leq t \leq 1$ and $f(\epsilon\omega'_0, t'_1\omega'_1, \dots, t'_k\omega'_k)$ for $\epsilon_0 t_0 \leq \epsilon \leq 1$. Therefore ω' can be joined to $f(t_1\omega^*(0))$. Similarly, ω'' can be joined to $f(t_1\omega^*(1))$. Finally, ω' can be joined to ω'' within $P^{(n)}[\mathcal{D}](U_0; U_1, \dots, U_k)$ by a continuous path. Thus $P^{(n)}[\mathcal{D}](U_0; U_1, \dots, U_k)$ is connected. ∎

For any $\epsilon > 0$ let $\Omega^\epsilon(p,n)$ be the set of translation structures in $\Omega(p,n)$ admitting a saddle connection of length at most ϵ. Further, for any $\kappa \geq \epsilon$ let $\Omega^{\epsilon,\kappa}(p,n)$ be the set of translation structures admitting a saddle connection of length at most ϵ and no nonhomologous saddle connections of length at most κ. Clearly, $\Omega^{\epsilon,\kappa}(p,n) \subset \Omega^\epsilon(p,n)$ and both sets are invariant under the $H(p,n)$ action. Let $\mathcal{Q}^\epsilon(p,n)$ and $\mathcal{Q}^{\epsilon,\kappa}(p,n)$ denote the subsets of $\mathcal{Q}(p,n)$ corresponding to $\Omega^\epsilon(p,n)$ and $\Omega^{\epsilon,\kappa}(p,n)$. Let $\mathcal{MQ}^\epsilon(p,n)$ and $\mathcal{MQ}^{\epsilon,\kappa}(p,n)$ denote the corresponding subsets of $\mathcal{MQ}(p,n)$. By $\mathcal{Q}_{\leq 1}(p,n)$ denote the set of $\omega \in \mathcal{Q}(p,n)$ such that $a(\omega) \leq 1$. The set $\mathcal{MQ}_{\leq 1}(p,n) \subset \mathcal{MQ}(p,n)$ is defined in a similar way. Now we let $\mathcal{Q}^\epsilon_{\leq 1}(p,n) = \mathcal{Q}^\epsilon(p,n) \cap \mathcal{Q}_{\leq 1}(p,n)$, $\mathcal{Q}^{\epsilon,\kappa}_{\leq 1}(p,n) = \mathcal{Q}^{\epsilon,\kappa}(p,n) \cap \mathcal{Q}_{\leq 1}(p,n)$, $\mathcal{MQ}^\epsilon_{\leq 1}(p,n) = \mathcal{MQ}^\epsilon(p,n) \cap \mathcal{MQ}_{\leq 1}(p,n)$, $\mathcal{MQ}^{\epsilon,\kappa}_{\leq 1}(p,n) = \mathcal{MQ}^{\epsilon,\kappa}(p,n) \cap \mathcal{MQ}_{\leq 1}(p,n)$. Similarly, we define sets $\mathcal{Q}^\epsilon_1(p,n)$, $\mathcal{Q}^{\epsilon,\kappa}_1(p,n)$, $\mathcal{MQ}^\epsilon_1(p,n)$, and $\mathcal{MQ}^{\epsilon,\kappa}_1(p,n)$.

Theorem 8.5 ([MS]) *Let μ_0 and μ denote the canonical measures on $\mathcal{MQ}_1(p,n)$ and $\mathcal{MQ}(p,n)$, respectively. There exists $c_{p,n} > 0$ such that $\mu_0(\mathcal{MQ}^\epsilon_1(p,n)) \leq c_{p,n}\epsilon^2$ and $\mu(\mathcal{MQ}^\epsilon_{\leq 1}(p,n)) \leq c_{p,n}\epsilon^2$ for any $\epsilon > 0$, and $\mu_0(\mathcal{MQ}^\epsilon_1(p,n) \setminus \mathcal{MQ}^{\epsilon,\kappa}_1(p,n)) \leq c_{p,n}\epsilon^2\kappa^2$ and $\mu(\mathcal{MQ}^\epsilon_{\leq 1}(p,n) \setminus \mathcal{MQ}^{\epsilon,\kappa}_{\leq 1}(p,n)) \leq c_{p,n}\epsilon^2\kappa^2$ for any $\kappa \geq \epsilon$.*

For any Delaunay triangulation piece $M(\tau, \prec) \subset \mathcal{Q}(p,n)$ we define a canonical ordered basis $\Gamma = (\gamma_1, \dots, \gamma_{2p+n-1})$ for the group $H_1(M_p, Z_n; \mathbb{Z})$. Namely, for any j, $1 \leq j \leq 2p+n-1$, γ_j is the least (with respect to $\prec$) homology class of an edge of τ that is not a linear combination of $\gamma_1, \dots, \gamma_{j-1}$. The homology classes $\gamma_1, \dots, \gamma_{2p+n-1}$ are determined up to multiplying by ± 1. By X_1 denote the set of vectors $v \in \mathbb{R}^2$ such that $\mathrm{hol}_\omega(\gamma_1) = v$ for some $\omega \in M(\tau, \prec) \cap \mathcal{Q}_{\leq 1}(p,n)$. Given vectors $v_1, \dots, v_{k-1} \in \mathbb{R}^2$, $1 < k \le 2p+n-1$, let $X_k(v_1, \dots, v_{k-1})$ denote the set of $v \in \mathbb{R}^2$ such that $\mathrm{hol}_\omega(\gamma_1) = v_1, \dots, \mathrm{hol}_\omega(\gamma_{k-1}) = v_{k-1}$, and $\mathrm{hol}_\omega(\gamma_k) = v$ for some $\omega \in M(\tau, \prec) \cap \mathcal{Q}_{\le 1}(p,n)$.

Theorem 8.6 ([MS]) *X_1 is contained in the disk $B(\sqrt{8/\pi})$. Any $X_k(v_1, \dots, v_{k-1})$ is contained in the union of $B(\sqrt{8/\pi})$ and finitely many rectangles of area 1, where the number of rectangles is bounded by a constant depending on $M(\tau, \prec)$.*

It is easy to see that a shortest saddle connection L_0 of a translation surface M is a Delaunay edge. If all shortest saddle connections of M are homologous to L_0, then any shortest saddle connection of those nonhomologous to L_0 is also a Delaunay edge. In view of this remark, Lemma 6.5, and Fubini's theorem, Theorem 8.5 is a corollary of Theorem 8.6. Further notice that $\mathcal{MQ}_1^\epsilon(p,n) = \mathcal{MQ}_1(p,n)$ for $\epsilon \ge \sqrt{8/\pi}$. Hence Theorem 6.6 is a corollary of Theorem 8.5. Theorems 6.6, 8.5, and 8.6 were proved in Section 10 of the paper [MS] (although only the first of them was explicitly formulated in [MS]).

Let $\tau \in \mathcal{T}(p,n)$ and e be an edge of τ. Suppose $\omega \in N(\tau)$. Pick a translation structure ω' in the isotopy class ω and let τ' be the triangulation by disjoint saddle connections of ω' such that $(\omega', \tau') \in \tau$. Let e' be the edge of τ' corresponding to e. By $\theta_{e,\tau}(\omega)$ denote the sum of two angles opposite e' in two triangles of τ' bounded by e'. It follows from the proof of Lemma 6.5 that $\theta_{e,\tau}(\omega)$ is well defined and depends continuously on ω. For any $\delta > 0$ let $M_\delta(\tau)$ denote the set of $\omega \in N(\tau)$ such that $\theta_{e,\tau}(\omega) < \pi - \delta$ for every edge e of τ. By Proposition 5.3, $M_\delta(\tau) \subset M(\tau)$ and $M(\tau) = \bigcup_{\delta > 0} M_\delta(\tau)$.

Suppose $\tau \in \mathcal{T}(p,n)$ and γ is an edge of τ. For any $\epsilon, \delta > 0$ and $\kappa > \epsilon$ let $S(\tau, \gamma; \epsilon, \kappa, \delta)$ denote the set of $\omega \in \mathcal{Q}_{\le 1}^{\epsilon,\kappa}(p,n) \cap M(\tau)$ such that $|\,\mathrm{hol}_\omega(\gamma)| \le \epsilon$ and for any edge e of τ we have $\theta_{e,\tau}(\omega) < \pi - \delta$ unless two triangles of τ bounded by e have two common edges while their third edges are homologous to γ. By $S(\tau; \epsilon, \kappa, \delta)$ denote the union of the sets $S(\tau, \gamma; \epsilon, \kappa, \delta)$ over all edges γ of τ.

Lemma 8.7 *Given $\Delta > 0$, there exist $\kappa, \delta > 0$ such that*

$$\tilde{\mu}(\mathcal{Q}_{\le 1}^\epsilon(p,n) \cap M(\tau)) - \tilde{\mu}(S(\tau; \epsilon, \kappa, \delta)) \le \Delta \epsilon^2$$

for all $\epsilon < \kappa$, where $\tilde{\mu}$ denotes the canonical measure on $\mathcal{Q}(p,n)$.

Proof. By Lemma 6.5, each Delaunay triangulation halfpiece $M^h(\tau, \prec)$ projects injectively to $\mathcal{MQ}(p,n)$. Up to a set of zero volume, $M(\tau)$ is the union of finitely many halfpieces. Hence Theorem 8.5 implies there exists $c_\tau > 0$ such that $\tilde{\mu}(\mathcal{Q}_{\le 1}^\epsilon(p,n) \cap M(\tau)) - \tilde{\mu}(\mathcal{Q}_{\le 1}^{\epsilon;\kappa}(p,n) \cap M(\tau)) \le c_\tau \epsilon^2 \kappa^2$ for $\kappa \ge \epsilon > 0$. Choose $\kappa > 0$ such that $c_\tau \kappa^2 \le \Delta/2$.

Suppose ω' is a translation structure in a class $\omega \in M(\tau) \cap \mathcal{Q}_{\le 1}^{\epsilon,\kappa}(p,n)$, $\epsilon < \kappa$, and let γ be an edge of τ such that $|\,\mathrm{hol}_\omega(\gamma)| \le \epsilon$. Given a Delaunay cell T of ω', there exist $d > 0$ and a map $\imath : B(d) \to M_p$ such that $\imath$ is a translation with respect to ω' and $T = \imath(T')$, where T' is the interior of a plane triangle inscribed in $B(d)$. Note that d is the distance from the point $q = \imath(0)$ to the set of singular points of ω'. By Lemma 5.4, if $d \geq \sqrt{2/\pi}$ then q belongs to a periodic cylinder C of length at most d^{-1}. It is easy to see that if the triangle T is disjoint from C then all edges of T are not longer than the length of C. As $\omega \in \mathcal{Q}^{\epsilon,\kappa}(p,n)$, the length of C is either at most ϵ or at least κ. In the former case, C is a regular cylinder bounded by saddle connections homologous to γ and $T \subset C$. In the latter case, $d \leq \kappa^{-1}$ and all edges of T are of length at most $2\kappa^{-1}$.

Suppose $x_1, x_2, x_3 \in \mathbb{R}^2$ are vertices of a triangle. For any $\delta > 0$ let $\Pi(x_1, x_2, x_3; \delta)$ be the set of $x \in \mathbb{R}^2$ such that $\pi - \delta < \angle x_1 x x_2 + \angle x_1 x_3 x_2 < \pi$ and x is separated from x_3 by the straight line passing through x_1 and x_2. $\Pi(x_1, x_2, x_3; \delta)$ is a domain bounded by two circles intersecting at x_1 and x_2, one of them being circumscribed about the triangle $x_1 x_2 x_3$. If we fix the length l of the segment $x_1 x_2$, the radius r of the circle passing through x_1, x_2, x_3, and δ, then the area of $\Pi(x_1, x_2, x_3; \delta)$ is bound to take at most two values. Let $\alpha(l, r, \delta)$ denote either of them. It is easy to see that for any $l_0, r_0 > 0$ we have $\sup_{l \geq l_0} \sup_{r \leq r_0} \alpha(l, r, \delta) \to 0$ as $\delta \to 0$.

Let γ and e be edges of τ. If two triangles bounded by e have two common edges while their third edges are homologous to γ, then for any $\omega \in N(\tau)$ the edge e crosses a cylinder of periodic geodesics of ω homologous to γ. Assume this is not the case. For $0 < \epsilon < \kappa$ and $\delta > 0$ let $S'(\tau, \gamma, e; \epsilon, \kappa, \delta)$ denote the set of $\omega \in M(\tau) \cap \mathcal{Q}^{\epsilon,\kappa}_{\leq 1}(p,n)$ such that $|\operatorname{hol}_\omega(\gamma)| \leq \epsilon$ and $\theta_{e,\tau}(\omega) > \pi - \delta$. If e is homologous to γ then $\theta_{e,\tau}(\omega) \leq 2\pi/3$ for all $\omega \in S'(\tau, \gamma, e; \epsilon, \kappa, \delta)$ since the angle opposite the shortest side of a triangle does not exceed $\pi/3$. Hence $S'(\tau, \gamma, e; \epsilon, \kappa, \delta)$ is empty for $\delta \leq \pi/3$. Now suppose e and γ are not homologous. Let T_1 and T_2 be triangles of τ bounded by e. Let e_1 be a side of T_1 different from e and e_2 be a side of T_2 different from e and not homologous to e_1. If an edge of T_1 or T_2 is homologous to γ, it is no loss to assume e_1 is homologous to γ. Suppose $\omega \in S'(\tau, \gamma, e; \epsilon, \kappa, \delta)$. Once $\operatorname{hol}_\omega(e)$ and $\operatorname{hol}_\omega(e_1)$ are fixed, the holonomy vector $\operatorname{hol}_\omega(e_2)$ belongs to a set $\Pi(x_1, x_2, x_3; \delta)$ of area $\alpha(l, r, \delta)$, where $l \geq \kappa$ and $r \leq r_1 = \max(\sqrt{2/\pi}, \kappa^{-1})$. If any edge of T_1 is homologous to an edge of T_2, we can say more (cf. the proof of Lemma 6.5). Namely, once $\operatorname{hol}_\omega(e)$ is fixed, the holonomy vector $\operatorname{hol}_\omega(e_2)$ belongs to a set $\Pi(x_1, x_2, x_3; \delta/2)$ of area $\alpha(l, l/2, \delta/2)$, where $l = |\operatorname{hol}_\omega(e)|$. By Lemma 6.2, there is a sequence $\gamma_1, \dots, \gamma_{2p+n-1}$ of edges of τ whose homology classes comprise a basis for $H_1(M_p, Z_n; \mathbb{Z})$. It can be assumed without loss of generality that this sequence contains an edge homologous to γ, the edges e, e_2, and e_1 unless the latter one is homologous to an edge of T_2. By the above $|\operatorname{hol}_\omega(\gamma_i)| \leq 2r_1$ unless γ_i crosses a cylinder of periodic geodesics homologous to γ. In the latter case we have $\operatorname{hol}_\omega(\gamma_i) = t \operatorname{hol}_\omega(\gamma) + v$, where $-1 < t < 1$, v is orthogonal to $\operatorname{hol}_\omega(\gamma)$, and $|v| \cdot |\operatorname{hol}_\omega(\gamma)| \leq 1$. It follows that $\operatorname{hol}_\omega(\gamma_i)$ belongs to a rectangle of area 4 depending on $\operatorname{hol}_\omega(\gamma)$. Since e_2 is not homologous to γ, Fubini's theorem implies $\tilde{\mu}(S'(\tau, \gamma, e; \epsilon, \kappa, \delta)) \leq \pi\epsilon^2 (4\pi r_1^2)^{m-2} \sup_{l \geq \kappa} \sup_{r \leq r_1} \alpha(l, r, \delta)$, where $m = 3(2p - 2 + n)$ is the number of edges of τ. For any $\delta' > \delta$ the set $M(\tau) \cap \mathcal{Q}^{\epsilon,\kappa}_{\leq 1}(p,n) \setminus S(\tau; \epsilon, \kappa, \delta)$ is contained in the union of at most m^2 sets of the form $S'(\tau, \gamma, e; \epsilon, \kappa, \delta')$. Therefore we can choose δ so that $\tilde{\mu}(M(\tau) \cap \mathcal{Q}^{\epsilon,\kappa}_{\leq 1}(p,n)) - \tilde{\mu}(S(\tau; \epsilon, \kappa, \delta)) \leq \Delta\epsilon^2/2$ for all $\epsilon < \kappa$. ■

Suppose ω is a translation structure. For any $\sigma \in [0, 1)$ and $R > 0$ let $N_0(\omega, \sigma, R)$ denote the number of periodic cylinders of ω of length at most R and of area greater than $\sigma a(\omega)$. By $s(\omega)$ denote the length of the shortest saddle connection of ω. Further, let $N_0^*(\omega,\sigma,R) = N_0(\omega,\sigma,s(\omega))$ if $R \ge s(\omega)$ and $N_0^*(\omega,\sigma,R) = 0$ if $0 < R < s(\omega)$. The numbers $N_0(\omega,\sigma,R)$, $N_0^*(\omega,\sigma,R)$, and $s(\omega)$ do not change when ω is replaced by an isomorphic structure, hence they are well defined for $\omega \in \mathcal{Q}(p,n)$ and $\omega \in \mathcal{MQ}(p,n)$.

Proposition 8.8 *Let $\mathcal{M}$ be a connected component of the moduli space $\mathcal{MQ}(p,n)$. By $\mathcal{M}_0$ denote the set of $\omega \in \mathcal{M}$ such that $a(\omega) \le 1$. Then there exists $c(\mathcal{M}) > 0$ such that for any $\sigma \in [0,1)$,*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^2} \int_{\mathcal{M}_0} N_0^*(\omega,\sigma,\epsilon)\, d\mu(\omega) = (1-\sigma)^{K-2} c(\mathcal{M}),$$

where K is half of the dimension of $\mathcal{M}$ and μ denotes the canonical measure on $\mathcal{M}$.

Proof. For any translation structure ω, $N_0^*(\omega,\sigma,\epsilon)$ is at most the number of saddle connections of length $s(\omega)$ of ω. As such saddle connections are Delaunay edges, they are disjoint by Proposition 5.2. Then Proposition 5.1 implies there exists $C > 0$ such that $N_0^*(\omega,\sigma,\epsilon) \le C$ for all $\omega \in \mathcal{MQ}(p,n)$.

By $\mathcal{M}_0'$ denote the subset of $\mathcal{M}_0$ corresponding to translation structures that have a regular periodic cylinder bounded by the shortest saddle connection. Clearly, $N_0^*(\omega,\sigma,\epsilon) = 0$ for $\omega \in \mathcal{M}_0 \setminus \mathcal{M}_0'$. Let $\pi_0 : \mathcal{Q}(p,n) \to \mathcal{MQ}(p,n)$ be the natural projection and $\tilde{\mu}$ be the canonical measure on $\mathcal{Q}(p,n)$. First consider the case $p = 1$. In this case $\mathcal{M}_0 = \mathcal{MQ}_{\le 1}(1,n)$. It is easy to see that the set $\pi_0(P^{(n)}(B(\epsilon) \cap \mathbb{R}^2_+))$ contains almost all elements of $\mathcal{M}_0' \cap \mathcal{MQ}^\epsilon(1,n)$ for any $\epsilon > 0$. There exists a mapping class $\phi \in \mathrm{Mod}(p,n)$ of order n such that $\phi\tau^{(n)} = \tau^{(n)}$ and ϕ sends $L^{(n)}$ to a homologous edge. Any set of the form $P^{(n)}(U)$ is invariant under ϕ. Suppose $\omega,\omega' \in P^{(n)}(B(\epsilon)\cap\mathbb{R}^2_+)\cap\mathcal{Q}^{\epsilon,\kappa}(p,n)$, $\kappa > \epsilon > 0$. Then $\pi_0(\omega) = \pi_0(\omega')$ only if $\omega' = \phi^m\omega$ for some m. Also, in this case $N_0^*(\omega,\sigma,\epsilon)$ equals the number of indices i such that $\omega \in P^{(n)}_{\sigma,i}(B(\epsilon) \cap \mathbb{R}^2_+)$. As shown in the proof of Lemma 8.2, $\tilde{\mu}(P^{(n)}_{\sigma,i}(B(\epsilon) \cap \mathbb{R}^2_+)) = (1-\sigma)^{n-1}\mathfrak{m}(B(\epsilon) \cap \mathbb{R}^2_+)c = (1-\sigma)^{n-1}\pi c\epsilon^2/2$, where $c > 0$. Since $\epsilon^{-2}\tilde{\mu}(P^{(n)}(B(\epsilon)) \setminus \mathcal{Q}^{\epsilon,\epsilon}(1,n)) \to 0$ as $\epsilon \to 0$, it follows that

$$\lim_{\epsilon\to 0} \epsilon^{-2} \int_{\mathcal{M}_0} N_0^*(\omega,\sigma,\epsilon)\, d\mu(\omega) = (1-\sigma)^{n-1}\pi c/2.$$

We proceed to the case $p > 1$. Consider an operation $f_0 = OPR[n_0;\mathcal{D}] = OPR[n_0;\mathcal{D}';\tau,\gamma]$, where $\tau \in \mathcal{T}(p,n)$ and γ is an edge of τ, and a family of sets $Y_0(\epsilon) = P^{(n_0)}[\mathcal{D}](\epsilon U_0; U_1,\dots,U_k)$, $\epsilon > 0$, where $U_0, U_1, \dots, U_k$ satisfy the assumptions of Lemma 8.4. Further assume that $U_0 \subset B(1)$ and $\tilde{\mu}(Y_0(\epsilon)\cap M(\tau))/\tilde{\mu}(Y_0(\epsilon)) \to 1$ as $\epsilon \to 0$. Clearly, $Y_0(\epsilon) \subset \mathcal{Q}^\epsilon_{\le 1}(p,n)$. Any translation structure in an isotopy class $\omega \in Y_0(\epsilon)$ has n_0 regular cylinders of periodic geodesics homologous to γ. Suppose $\omega \in \mathcal{Q}^{\epsilon,\kappa}_{\le 1}(p,n)$ for some $\kappa > \epsilon$. Then the cylinders homologous to γ are the only periodic cylinders of length $s(\omega)$. It follows that $N_0^*(\omega,0,\epsilon) = n_0$ while $N_0^*(\omega,\sigma,\epsilon)$ is equal to the number of indices i such that $\omega \in P^{(n_0)}_{\sigma,i}[\mathcal{D}](\epsilon U_0; U_1,\dots,U_k)$. Therefore

$$\Big|\int_{Y_0(\epsilon)} N_0^*(\omega,\sigma,\epsilon)\, d\tilde{\mu}(\omega) - \sum_{i=1}^{n_0} \tilde{\mu}(P^{(n_0)}_{\sigma,i}[\mathcal{D}](\epsilon U_0; U_1,\dots,U_k))\Big| \le$$

$$\max(C, n_0)\tilde{\mu}(Y_0(\epsilon) \setminus \mathcal{Q}^{\epsilon,\epsilon}_{\le 1}(p,n)).$$

Note that $\tilde{\mu}(M(\tau) \cap \mathcal{Q}^{\epsilon}_{\le 1}(p,n) \setminus \mathcal{Q}^{\epsilon,\epsilon}_{\le 1}(p,n)) \le c_\tau \epsilon^4$, where c_τ depends on τ (see the proof of Lemma 8.7). By assumption, $\tilde{\mu}(Y_0(\epsilon) \setminus M(\tau))/\tilde{\mu}(Y_0(\epsilon)) \to 0$ as $\epsilon \to 0$. Hence Lemma 8.4 implies there exists $c_0 > 0$ such that

$$\lim_{\epsilon \to 0} \epsilon^{-2} \int_{Y_0(\epsilon)} N_0^*(\omega, \sigma, \epsilon)\, d\tilde{\mu}(\omega) = (1-\sigma)^{K-2} c_0.$$

For any $\Delta > 0$ we wish to find finitely many maps $f_i = OPR[n_i; \mathcal{D}_i]$ and families of sets $Y_i(\epsilon) = P^{(n_i)}[\mathcal{D}_i](\epsilon U_{i0}; U_{i1}, \dots, U_{ik_i})$, $i = 1, \dots, m$, that satisfy the conditions imposed above on f_0 and $Y_0(\epsilon)$. Besides, we require that $\pi_0(Y_i(\epsilon)) \subset \mathcal{M}_0$, each $Y_i(\epsilon)$ be invariant under a mapping class $\phi_i \in \mathrm{Mod}(p,n)$ of finite order o_i, $\epsilon^{-2}|\mu(\pi_0(\cup_i Y_i(\epsilon))) - \sum_i o_i^{-1}\tilde{\mu}(Y_i(\epsilon))| \to 0$ as $\epsilon \to 0$, and $\mu(\mathcal{M}_0' \cap \mathcal{MQ}^{\epsilon}(p,n) \setminus \pi_0(\cup_i Y_i(\epsilon))) \le \Delta\epsilon^2$ for sufficiently small ϵ. First let us show how this helps to prove the proposition. By the above

$$\lim_{\epsilon \to 0} \epsilon^{-2} \int_{Y_i(\epsilon)} N_0^*(\omega, \sigma, \epsilon)\, d\tilde{\mu}(\omega) = (1-\sigma)^{K-2} c_i$$

for some $c_i > 0$. Given $\omega \in \mathcal{M}_0$, let $\omega \in Y'(\epsilon)$ if $\omega \in \pi_0(Y_i(\epsilon)) \cap \pi_0(Y_j(\epsilon))$ for some $i \ne j$, let $\omega \in Y''(\epsilon)$ if more than o_i elements of some $Y_i(\epsilon)$ are mapped to ω by π_0, and let $\omega \in Y'''(\epsilon)$ if $\omega \in \pi_0(Y_i(\epsilon))$ for some i but $\pi_0^{-1}(\omega) \cap Y_i(\epsilon)$ has less than o_i elements. Any translation structure in an isomorphy class $\omega \in Y'''(\epsilon)$ admits nontrivial automorphism, therefore $\mu(Y'''(\epsilon)) = 0$. Observe that $\mu(\pi_0(\cup_i Y_i(\epsilon))) + \mu(Y'(\epsilon)) \le \sum_i \mu(\pi_0(Y_i(\epsilon)))$ and $\sum_i \mu(\pi_0(Y_i(\epsilon))) + (\max_i o_i)^{-1}\mu(Y''(\epsilon)) \le \sum_i o_i^{-1}\tilde{\mu}(Y_i(\epsilon))$. The above assumptions imply $\epsilon^{-2}\mu(Y'(\epsilon) \cup Y''(\epsilon)) \to 0$ as $\epsilon \to 0$. Since $N_0^*(\omega, \sigma, \epsilon)$ is uniformly bounded and there are only finitely many families $Y_i(\epsilon)$, it follows that

$$\lim_{\epsilon \to 0} \epsilon^{-2} \int_{\pi_0(\cup_i Y_i(\epsilon))} N_0^*(\omega, \sigma, \epsilon)\, d\mu(\omega) = (1-\sigma)^{K-2} \sum_i o_i^{-1} c_i.$$

Moreover, $\pi_0(\cup_i Y_i(\epsilon)) \subset \mathcal{M}_0$ and

$$\int_{\mathcal{M}_0 \setminus \pi_0(\cup_i Y_i(\epsilon))} N_0^*(\omega, \sigma, \epsilon)\, d\mu(\omega) \le C\Delta\epsilon^2$$

for small ϵ. As Δ can be chosen arbitrarily small, the proposition follows.

It remains to fetch the needed maps f_i and sets $Y_i(\epsilon)$. The maps will be of the form $f_i = OPR[n_i; \mathcal{D}_i] = OPR[n_i; \widetilde{\mathcal{D}}_{i1}, \dots, \widetilde{\mathcal{D}}_{ik_i}; \tilde{l}_i; \tau_i, \gamma_i]$, $\widetilde{\mathcal{D}}_{ij} = (\tau_{ij}, e_{r,ij}, e_{l,ij}; \mathcal{D}_{ij})$, $\tilde{l}_i = (l_{i1}, \dots, l_{ik_i})$, where $\tau_i \in \mathcal{T}(p,n)$. Recall that γ_i is an oriented edge of τ_i, $e_{r,ij}$ and $e_{l,ij}$ are oriented edges of τ_{ij}, and $\tilde{l}_i$ is a sequence of integers. We require that if $\phi\tau_i = \tau_{i'}$ for some $\phi \in \mathrm{Mod}(p,n)$ then $\tau_i = \tau_{i'}$. Moreover, if $\phi\tau_i = \tau_{i'}$ and $\phi(\gamma_i)$ is homologous to $\gamma_{i'}$ then $\tau_i = \tau_{i'}$ and $\gamma_i = \gamma_{i'}$. If $\tau_{ij}, \tau_{i'j'} \in \mathcal{T}(p', n')$ and there exists $\phi \in \mathrm{Mod}(p', n')$ such that $\phi\tau_{ij} = \tau_{i'j'}$, $\phi(e_{r,ij}) = e_{r,i'j'}$, $\phi(e_{l,ij}) = e_{l,i'j'}$, then $\widetilde{\mathcal{D}}_{ij} = \widetilde{\mathcal{D}}_{i'j'}$. We assume $f_1, f_2, \dots$ is a maximal list of maps satisfying the above conditions. It is easy to see that the list is finite.

Suppose $\tau \in \mathcal{T}(p', n')$ and e_r, e_l are oriented edges of τ. Choose an inserting operation $f = INS[\tau, e_r, e_l; \mathcal{D}; \tau', \gamma']$. For any $v \in S^1$ and $\delta > 0$ we define $U_f(\tau, e_r, e_l; v, \delta)$ as the set of $\omega \in N(\tau)$ such that $f(tv, \omega) \in M_\delta(\tau')$ for all sufficiently small $t > 0$. Assume $f(tv, \omega)$ is defined for small $t > 0$, that is, pairs of vectors $(\mathrm{hol}_\omega(e_r), v)$ and $(\mathrm{hol}_\omega(e_l), v)$ induce the standard orientation in $\mathbb{R}^2$. Let β_r (resp. β_l) denote the angle between $\mathrm{hol}_\omega(e_r)$ (resp. $\mathrm{hol}_\omega(e_l)$) and v. Let T_r^+ be the triangle of τ such that e_r bounds T_r^+ and the orientation of e_r agrees with the counterclockwise orientation of the boundary of T_r^+. Let T_r^- be the other triangle of τ bounded by e_r. Let θ_r^+ and θ_r^- denote the measures relative to translation structures in the class ω of the angles of T_r^+ and T_r^- opposite e_r. Similarly, we introduce triangles T_l^+, T_l^- and angles θ_l^+, θ_l^-. In the case $e_r \neq e_l$, the set $U_f(\tau, e_r, e_l; v, \delta)$ does not depend on the choice of f. Namely, $\omega \in U_f(\tau, e_r, e_l; v, \delta)$ if and only if $\omega \in M_\delta(\tau)$ and $\beta_r + \theta_r^+$, $\pi - \beta_r + \theta_r^-$, $\pi - \beta_l + \theta_l^+$, $\beta_l + \theta_l^-$ are less than $\pi - \delta$. In the case $e_r = e_l$, there are two principal choices. The triangle T_r^+ corresponds to a triangle T' of τ'. Let e' be the edge of T' corresponding to e_r and T_0' be the triangle separated from T' by e'. Then γ' is an edge of T_0'. If T_0' lies to the right of γ' then $\omega \in U_f(\tau, e_r, e_l; v, \delta)$ if and only if $\omega \in M_\delta(\tau)$ and $\beta_r + \theta_r^+$, $\beta_r + \theta_r^-$, $2(\pi - \beta_r)$ are less than $\pi - \delta$. Otherwise $\omega \in U_f(\tau, e_r, e_l; v, \delta)$ if and only if $\omega \in M_\delta(\tau)$ and $\pi - \beta_r + \theta_r^+$, $\pi - \beta_r + \theta_r^-$, $2\beta_r$ are less than $\pi - \delta$. In particular, if the angle between $v, v' \in S^1$ does not exceed $\delta' < \delta$ then $U_f(\tau, e_r, e_l; v, \delta) \subset U_f(\tau, e_r, e_l; v', \delta - \delta')$.

Choose $\delta > 0$ such that π/δ is an integer and divide the semicircle $S^1 \cap \mathbb{R}^2_+$ into arcs of equal length δ. Suppose u is the interior of one of the arcs. Let $U_{0u} = \{tv \mid v \in u,\ 0 < t < 1\}$. By v_0 denote the midpoint of u. For $1 \leq j \leq k_i$ let f_{ij} denote the j^{th} inserting operation of those used when building f_i. Given $\kappa > 0$, let $U_{iju\kappa} = U_{f_{ij}}(\tau_{ij}, e_{r,ij}, e_{l,ij}; v_0, \delta) \cap \mathcal{Q}_1(p_{ij}, n_{ij}) \setminus \mathcal{Q}_1^\kappa(p_{ij}, n_{ij})$, where $\tau_{ij} \in \mathcal{T}(p_{ij}, n_{ij})$. Finally, let $Y_{iu\kappa}(\epsilon) = P^{(n_i)}[\mathcal{D}_i](\epsilon U_{0u}; U_{i1u\kappa}, \dots, U_{ik_iu\kappa})$, $\epsilon > 0$. Now we shall check whether the families of sets $Y_{iu\kappa}(\epsilon)$ satisfy the above conditions.

Suppose $\omega \in U_{iju\kappa}$. Any Delaunay cell of a translation structure in the class ω is isometric to the interior of a triangle inscribed in a circle of radius d. Since $\omega \in \mathcal{Q}_1(p_{ij}, n_{ij}) \setminus \mathcal{Q}_1^\kappa(p_{ij}, n_{ij})$, it follows from Lemma 5.4 that $d \leq d_0$, where $d_0 = \max(\sqrt{2/\pi}, \kappa^{-1})$. In particular, all Delaunay edges are of length at least κ and at most $2d_0$. Then each angle θ of a Delaunay triangle satisfies $\sin\theta \geq \kappa/(2d_0)$. Since $U_{iju\kappa} \subset M(\tau_{ij})$, it follows that the closure of $U_{iju\kappa}$ is a compact subset of $N(\tau_{ij})$. Note that $f_{ij} = INS[\widetilde{\mathcal{D}}_{ij}; \tau'_{ij}, \gamma'_{ij}]$ for some $\tau'_{ij} \in \mathcal{T}(p'_{ij}, n'_{ij})$. There exists $\epsilon_0 = \epsilon_0(\kappa, \delta) > 0$ such that $f_{ij}(v, \omega) \in M_{\delta/4}(\tau'_{ij}) \cap \mathcal{Q}^{\epsilon_0, \kappa/2}(p'_{ij}, n'_{ij})$ for all $v \in \epsilon_0 U_{0u}$ and $\omega \in U_{iju\kappa}$.

Suppose $\omega = GLU[\tilde{\tau}_1, \tilde{\gamma}_1, \dots, \tilde{\tau}_k, \tilde{\gamma}_k; \tilde{\tau}, \tilde{\gamma}](\omega_1, \dots, \omega_k)$ and $\tilde{\gamma}_j$ $(1 \leq j \leq k)$ is a unique shortest edge of $\tilde{\tau}_j$ relative to ω_j. Any edge e of $\tilde{\tau}_j$ different from $\tilde{\gamma}_j$ is assigned a unique edge $\tilde{e}$ of $\tilde{\tau}$ so that $\theta_{e,\tilde{\tau}_j}(\omega_j) = \theta_{\tilde{e},\tilde{\tau}}(\omega)$. Since the angle opposite the shortest side of a triangle does not exceed $\pi/3$, we have $\theta_{\tilde{\gamma},\tilde{\tau}}(\omega) \leq 2\pi/3$ and $\theta_{\tilde{\gamma}_j,\tilde{\tau}_j}(\omega_j) \leq 2\pi/3$. By Proposition 5.3, $\omega \in M(\tilde{\tau})$ if and only if $\omega_j \in M(\tilde{\tau}_j)$ for $1 \leq j \leq k$. Further suppose $\omega' = GLU[\tilde{\tau}'_1, \tilde{\gamma}'_1, \dots, \tilde{\tau}'_{k'}, \tilde{\gamma}'_{k'}; \tilde{\tau}', \tilde{\gamma}'](\omega'_1, \dots, \omega'_k)$, where $\tilde{\gamma}'_j$ is a unique shortest edge of $\tilde{\tau}'_j$ relative to ω'_j. Assume that $\omega \in M(\tilde{\tau})$, $\omega' \in M(\tilde{\tau}')$, and $\mathrm{hol}_\omega(\tilde{\gamma}), \mathrm{hol}_{\omega'}(\tilde{\gamma}') \in \mathbb{R}^2_+$. Then ω is isomorphic to ω' if and only if $k = k'$ and for some $1 \le k_0 \le k$ the classes $\omega_1, \dots, \omega_k$ are isomorphic to $\omega'_{k_0}, \dots, \omega'_k, \omega'_1, \dots, \omega'_{k_0-1}$, respectively.

All mapping classes $\phi_i \in \mathcal{T}(p,n)$ such that $\phi_i \tau_i = \tau_i$ and ϕ_i sends γ_i to a homologous edge form a cyclic subgroup. Assume ϕ_i generates this subgroup. The order o_i of ϕ_i divides k_i and n_i. For any $\omega = f_i(\omega_0, \omega_1, \dots, \omega_{k_i})$ one has $\omega\phi_i^{-1} = f_i(\omega_0\phi^{-1}, \omega_k, \dots \omega_{k_i}, \omega_1, \dots, \omega_{k-1})$, where $1 \le k \le k_i$ and $\phi \in \mathrm{Mod}(1, n_i)$ is such that $\phi\tau^{(n_i)} = \tau^{(n_i)}$ and $\phi(L^{(n_i)})$ is homologous to $L^{(n_i)}$. It follows that each $Y_{iu\kappa}(\epsilon)$ is invariant under ϕ_i.

Let $\omega = f_{ij}(v, \omega_0)$, $\omega' = f_{i'j'}(v', \omega'_0)$, where $\omega_0 \in M(\tau_{ij})$, $\omega'_0 \in M(\tau_{i'j'})$, and $v, v' \in \mathbb{R}^2_+$. Suppose $|v|, |v'| \le \epsilon$ and $\omega \in M(\tau'_{ij}) \cap \mathcal{Q}^{\epsilon,\kappa'}(p'_{ij}, n'_{ij})$, $\omega' \in M(\tau'_{i'j'}) \cap \mathcal{Q}^{\epsilon,\kappa'}(p'_{i'j'}, n'_{i'j'})$ for some $\kappa' > \epsilon$. Then ω is isomorphic to ω' if and only if $i = i'$, $j = j'$, $v = v'$, and $\omega_0 = \omega'_0$. Now let $\omega \in Y_{iu\kappa}(\epsilon)$, $\omega' \in Y_{i'u'\kappa}(\epsilon)$. Suppose $\omega \in M(\tau_i) \cap \mathcal{Q}^{\epsilon,\kappa'}(p,n)$ and $\omega' \in M(\tau_{i'}) \cap \mathcal{Q}^{\epsilon,\kappa'}(p,n)$ for some $\kappa' > \epsilon$. Then it follows from the above that $\pi_0(\omega) = \pi_0(\omega')$ only if $i = i'$, $u = u'$, and ω is mapped to ω' by an iterate of ϕ_i.

Using techniques of the proof of Lemma 8.4, we derive from the above that as $\epsilon \to 0$, $\tilde\mu(Y_{iu\kappa}(\epsilon) \cap M(\tau_i))/\tilde\mu(Y_{iu\kappa}(\epsilon)) \to 1$, $\epsilon^{-2}|\tilde\mu(Y_{iu\kappa}(\epsilon)) - o_i\mu(\pi_0(Y_{iu\kappa}(\epsilon)))| \to 0$, and $\epsilon^{-2}\mu(\pi_0(Y_{iu\kappa}(\epsilon)) \cap \pi_0(Y_{i'u'\kappa}(\epsilon))) \to 0$ unless $i = i'$ and $u = u'$.

Suppose $\omega \in S(\tau, \gamma; \epsilon, 2\kappa, 2\delta)$ for some $\tau \in \mathcal{T}(p,n)$. As $S(\tau, \gamma; \epsilon, 2\kappa, 2\delta)$ does not depend on the orientation of γ, we orient γ so that $\mathrm{hol}_\omega(\gamma)$ belongs to the closure of $\mathbb{R}^2_+$. Then $\mathrm{hol}_\omega(\gamma)$ is in the closure of ϵU_{0u} for an arc u. Assume $\mathrm{hol}_\omega(\gamma) \in \epsilon U_{0u}$. Note that this assumption holds for almost all elements of $S(\tau, \gamma; \epsilon, 2\kappa, 2\delta)$. First we apply a cutting operation to ω and γ. Let $(\omega'_1, \dots, \omega'_k) = CUT[\tau, \gamma; \tau'_1, \gamma'_1, \dots, \tau'_k, \gamma'_k](\omega)$, where $\tau'_j \in \mathcal{T}(p'_j, n'_j)$, $1 \le j \le k$. Clearly, each τ'_j has no edge homologous to γ'_j. In the case $p'_j = n'_j = 1$, it is no loss to assume that $\tau'_j = \tau^{(1)}$, $\gamma'_j = L^{(1)}$; then $\omega'_j \in P^{(1)}$. If none of ω'_j belongs to $\mathcal{Q}(1,1)$ then $\pi_0(\omega) \notin \mathcal{M}'_0$. Assume this is not the case. Choose j such that $(p'_j, n'_j) \ne (1,1)$ (this is possible as $p > 1$). Then $\omega'_j \in S(\tau'_j, \gamma'_j; \epsilon, 2\kappa, 2\delta) \subset M_{2\delta}(\tau'_j) \cap \mathcal{Q}^{\epsilon,2\kappa}_{\le 1}(p'_j, n'_j)$ assuming $2\delta < \pi/3$. There exists $\epsilon_1 = \epsilon_1(\kappa, \delta) > 0$ such that any collapsing operation of the form $COL[\tau'_j, \gamma'_j; \mathcal{D}'; \tilde\tau, \tilde e_r, \tilde e_l]$ is defined on $S(\tau'_j, \gamma'_j; \epsilon, 2\kappa, 2\delta)$ for $\epsilon \le \epsilon_1$. If $\tilde\omega = COL[\tau'_j, \gamma'_j; \mathcal{D}'; \tilde\tau, \tilde e_r, \tilde e_l](\omega'_j)$ then $\omega'_j = g(\mathrm{hol}_\omega(\gamma), \tilde\omega)$, where $g = INS[\tilde\tau, \tilde e_r, \tilde e_l; \mathcal{D}; \tau'_j, \gamma'_j]$ for a set $\mathcal{D}$ of parameters. Consequently, ω can be obtained by an operation $OPR[\widetilde{\mathcal{D}}; \tau, \gamma]$. Assuming ϵ is small enough, one has $\tilde\omega \in U_g(\tilde\tau, \tilde e_r, \tilde e_l; v_0, \delta)$, where v_0 is the midpoint of u. Also, $\tilde\omega = t\tilde\omega_0$, where $t > 0$, $a(\tilde\omega_0) = 1$, and $s(\tilde\omega_0) > \kappa$. Note that $\mathcal{D}$ depends on $\mathcal{D}'$, moreover, any $\mathcal{D}$ corresponds to some $\mathcal{D}'$. It follows that we can choose the cutting and the collapsing operations so that $g = f_{i'j'}$ for some i', j'. Then $\tilde\omega_0 \in U_{i'j'u\kappa}$. The conditions imposed above on $f_1, f_2, \dots$ imply that $\omega\phi^{-1} \in Y_{iu\kappa}(\epsilon)$ for some i and $\phi \in \mathrm{Mod}(p,n)$ such that $\phi\tau = \tau_i$ and $\phi(\gamma)$ is homologous to γ_i. Thus for small ϵ the set $\pi_0(S(\tau, \gamma; \epsilon, 2\kappa, 2\delta)) \cap \mathcal{M}'_0$ is contained in $\pi_0(\cup_{i,u} Y_{iu\kappa}(\epsilon))$ up to a subset of zero volume. Since the $\mathrm{Mod}(p,n)$ action on $\mathcal{T}(p,n)$ has only finitely many orbits, it follows from Lemma 8.7 that we can choose κ and δ so that $\mu(\mathcal{M}'_0 \cap \mathcal{MQ}^\epsilon(p,n) \setminus \pi_0(\cup_{i,u} Y_{iu\kappa}(\epsilon))) \le \Delta\epsilon^2$ for small ϵ.

The only condition not verified yet is $\pi_0(Y_{iu\kappa}(\epsilon)) \subset \mathcal{M}_0$. It surely holds when the moduli space $\mathcal{MQ}(p,n)$ is connected. If this is not the case, we have to modify the construction slightly. Suppose U'_j, $1 \le j \le k_i$, is a connected component of the open set $U_{iju\kappa}$. Then sets $Y(\epsilon) = P^{(n_i)}[\mathcal{D}_i](\epsilon U_{0u}; U'_1, \dots, U'_{k_i})$ are connected by Lemma 8.4. Hence sets $\pi_0(Y(\epsilon))$ are either contained in $\mathcal{M}_0$ or disjoint from $\mathcal{M}_0$. Clearly, $Y_{iu\kappa}(\epsilon)$ is the disjoint union of at most countably many sets of the form $Y(\epsilon)$. The mapping class ϕ_i permutes these sets. Now we replace $Y_{iu\kappa}(\epsilon)$ by a finite number of subsets of the form $Y(\epsilon)$. For any $\Delta > 0$ this finite number can be chosen so that the union of the other subsets has volume at most $\Delta\epsilon^2$. Then we discard sets whose projections to $\mathcal{MQ}(p,n)$ are disjoint from $\mathcal{M}_0$. Also, among sets $Y(\epsilon), Y(\epsilon)\phi_i^{-1}, Y(\epsilon)\phi_i^{-2}, \dots$ we discard all but one. The construction is completed. ∎

Proof of Proposition 7.4. Let $\mathcal{C}$ be a connected component of $\mathcal{MQ}_1(p,n)$ and μ_0 be the canonical measure on $\mathcal{C}$. The formula (11) can be rewritten as follows:

$$b(\sigma,\epsilon) = \frac{1}{\mu_0(\mathcal{C})} \int_{\mathcal{C}} N_0(\omega,\sigma,\epsilon)\, d\mu_0(\omega).$$

Besides, let

$$b^*(\sigma,\epsilon) = \frac{1}{\mu_0(\mathcal{C})} \int_{\mathcal{C}} N_0^*(\omega,\sigma,\epsilon)\, d\mu_0(\omega).$$

First we estimate the difference $b(\sigma,\epsilon) - b^*(\sigma,\epsilon)$. Clearly, $N_0(\omega,0,\epsilon) \ge N_0(\omega,\sigma,\epsilon) \ge N_0^*(\omega,\sigma,\epsilon) \ge 0$ for all $\omega \in \mathcal{C}$. If $\omega \notin \mathcal{MQ}_1^{\epsilon}(p,n)$ then $N_0(\omega,\sigma,\epsilon) = 0$. If $\omega \in \mathcal{MQ}_1^{\epsilon,\kappa}(p,n)$ for some $\kappa > \epsilon$, then $N_0(\omega,\sigma,\epsilon) = N_0^*(\omega,\sigma,\epsilon)$. It follows that for any $\kappa > \epsilon$,

$$0 \le b(\sigma,\epsilon) - b^*(\sigma,\epsilon) \le \frac{1}{\mu_0(\mathcal{C})} \int_{\mathcal{C}\cap\mathcal{MQ}_1^{\epsilon}(p,n)\setminus\mathcal{MQ}_1^{\epsilon,\kappa}(p,n)} N_0(\omega,0,\epsilon)\, d\mu_0(\omega).$$

By Theorem 7.3, there exist $c_0, \kappa_0 > 0$ such that $N_0(\omega,0,\epsilon) \le c_0(\epsilon/s(\omega))^{3/2}$ for all $\omega \in \mathcal{C}$ and $\epsilon < \kappa_0$. Suppose $\omega \in \mathcal{C} \cap \mathcal{MQ}_1^{\epsilon}(p,n) \setminus \mathcal{MQ}_1^{\epsilon,\kappa}(p,n)$, where $\epsilon < \kappa \le \kappa_0$. Obviously, $s(\omega) \le \epsilon$, hence $4^{-k-1}\epsilon < s(\omega) \le 4^{-k}\epsilon$ for an integer $k \ge 0$. Then $\omega \in \mathcal{MQ}_1^{4^{-k}\epsilon}(p,n) \setminus \mathcal{MQ}_1^{4^{-k}\epsilon,\kappa}(p,n)$ and $N_0(\omega,0,\epsilon) \le (4^{k+1})^{3/2}c_0$. It follows that

$$b(\sigma,\epsilon) - b^*(\sigma,\epsilon) \le \frac{c_0}{\mu_0(\mathcal{C})} \sum_{k=0}^{\infty} 8^{k+1} \mu_0(\mathcal{MQ}_1^{4^{-k}\epsilon}(p,n) \setminus \mathcal{MQ}_1^{4^{-k}\epsilon,\kappa}(p,n))$$

provided $\epsilon < \kappa \le \kappa_0$. Furthermore, Theorem 8.5 implies that

$$b(\sigma,\epsilon) - b^*(\sigma,\epsilon) \le \frac{c_0 c_{p,n}}{\mu_0(\mathcal{C})} \sum_{k=0}^{\infty} 8^{k+1} 4^{-2k} \epsilon^2\kappa^2 = \frac{16 c_0 c_{p,n}}{\mu_0(\mathcal{C})}\, \epsilon^2\kappa^2,$$

where $c_{p,n} > 0$ is a constant. As κ can be chosen arbitrarily small, it follows that

$$\lim_{\epsilon\to 0} \epsilon^{-2}(b(\sigma,\epsilon) - b^*(\sigma,\epsilon)) = 0.$$

Now we estimate $b^*(\sigma,\epsilon)$. By $\mathcal{C}_\infty$ denote the connected component of $\mathcal{MQ}(p,n)$ containing $\mathcal{C}$. Clearly, $\mathcal{C}_\infty = \{t\omega \mid \omega \in \mathcal{C},\ t > 0\}$. For any $t > 0$ let $\mathcal{C}_t$ denote the set of $\omega \in \mathcal{C}_\infty$ such that $a(\omega) \leq t$. Let μ be the canonical measure on $\mathcal{C}_\infty$. For all $\omega \in \mathcal{C}$ and $t > 0$ we have $a(t\omega) = t^2$, $s(t\omega) = t \cdot s(\omega)$, and $N_0^*(\omega, \sigma, \epsilon) = N_0^*(t\omega, \sigma, t\epsilon)$. It follows that

$$b^*(\sigma, \epsilon) = \frac{1}{\mu(\mathcal{C}_1)} \int_{\mathcal{C}_1} N_0^*(\omega, \sigma, \epsilon\sqrt{a(\omega)})\, d\mu(\omega).$$

For any $\sigma \in [0, 1)$ and positive ϵ, t, α let

$$b_0(\sigma, \epsilon; t, \alpha) = \int_{\mathcal{C}_t} N_0^*(\omega, \sigma, \epsilon\sqrt{\alpha})\, d\mu(\omega).$$

Denote by K half of the dimension of $\mathcal{C}_\infty$. By Proposition 8.8, there exists $c > 0$ such that $\epsilon^{-2} b_0(\sigma, \epsilon; 1, 1) \to (1-\sigma)^{K-2} c$ as $\epsilon \to 0$. For arbitrary t and α we have

$$b_0(\sigma, \epsilon; t, \alpha) = t^K \int_{\mathcal{C}_1} N_0^*(t^{1/2}\omega, \sigma, \epsilon\sqrt{\alpha})\, d\mu(\omega) = t^K b_0(\sigma, \epsilon\sqrt{\alpha/t}; 1, 1),$$

therefore

$$\lim_{\epsilon \to 0} \epsilon^{-2} b_0(\sigma, \epsilon; t, \alpha) = t^{K-1} \alpha (1-\sigma)^{K-2} c.$$

Suppose $0 < t_1 < t_2$. Then

$$b_0(\sigma, \epsilon; t_2, t_1) - b_0(\sigma, \epsilon; t_1, t_1) \leq$$

$$\int_{\mathcal{C}_{t_2} \setminus \mathcal{C}_{t_1}} N_0^*(\omega, \sigma, \epsilon\sqrt{a(\omega)})\, d\mu(\omega) \leq b_0(\sigma, \epsilon; t_2, t_2) - b_0(\sigma, \epsilon; t_1, t_2).$$

It follows that for any integer $k > 0$ we have $b_k^-(\sigma, \epsilon) \leq \mu(\mathcal{C}_1) b^*(\sigma, \epsilon) \leq b_k^+(\sigma, \epsilon)$, where

$$b_k^-(\sigma, \epsilon) = \sum_{i=1}^{k} \Big(b_0(\sigma, \epsilon; i/k, (i-1)/k) - b_0(\sigma, \epsilon; (i-1)/k, (i-1)/k) \Big),$$

$$b_k^+(\sigma, \epsilon) = \sum_{i=1}^{k} \Big(b_0(\sigma, \epsilon; i/k, i/k) - b_0(\sigma, \epsilon; (i-1)/k, i/k) \Big).$$

By the above,

$$\lim_{\epsilon \to 0} \epsilon^{-2} b_k^-(\sigma, \epsilon) = (1-\sigma)^{K-2} c \sum_{i=1}^{k} \frac{i-1}{k} \Big(\Big(\frac{i}{k}\Big)^{K-1} - \Big(\frac{i-1}{k}\Big)^{K-1} \Big),$$

$$\lim_{\epsilon \to 0} \epsilon^{-2} b_k^+(\sigma, \epsilon) = (1-\sigma)^{K-2} c \sum_{i=1}^{k} \frac{i}{k} \Big(\Big(\frac{i}{k}\Big)^{K-1} - \Big(\frac{i-1}{k}\Big)^{K-1} \Big).$$

In particular, $\lim_{\epsilon \to 0} \epsilon^{-2} (b_k^+(\sigma, \epsilon) - b_k^-(\sigma, \epsilon)) = k^{-1} (1-\sigma)^{K-2} c$. As k can be chosen arbitrarily large, this implies that

$$\lim_{\epsilon \to 0} \epsilon^{-2} b^*(\sigma, \epsilon) = (1-\sigma)^{K-2} \frac{c}{\mu(\mathcal{C}_1)} \int_0^1 t\, dt^{K-1}.$$

Since $\epsilon^{-2}(b(\sigma,\epsilon) - b^*(\sigma,\epsilon)) \to 0$ as $\epsilon \to 0$, it follows that

$$\lim_{\epsilon\to 0} \frac{b(\sigma,\epsilon)}{b(0,\epsilon)} = \lim_{\epsilon\to 0} \frac{b^*(\sigma,\epsilon)}{b^*(0,\epsilon)} = (1-\sigma)^{K-2}.$$

It remains to notice that $K = 2p + n - 1$. ∎

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